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SOLUTIONS TO CUMULATIVE EXERCISES

A Supplement to A Foundation for Implementation

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Exercise 1: Degree and Radian Measure

1. a) $\frac{25}{180} = \frac{x}{\pi} \Rightarrow x = \frac{25\pi}{180} = \frac{5\pi}{36}$ b) $\frac{-120}{180} = \frac{x}{\pi} \Rightarrow x = \frac{-120\pi}{180} = \frac{-2\pi}{3}$

c) $\frac{460}{180} = \frac{x}{\pi} \Rightarrow x = \frac{460\pi}{180} = \frac{23\pi}{9}$ d) $\frac{330}{180} = \frac{x}{\pi} \Rightarrow x = \frac{330\pi}{180} = \frac{11\pi}{6}$

2. a) $\frac{-7\pi/6}{\pi} = \frac{x}{180} \Rightarrow x = \frac{-7}{6}(180) = -210°$ c) $\frac{2.634}{\pi} = \frac{x}{180} \Rightarrow x = \frac{2.634(180)}{\pi}$

$= 150.9°$

b) $\frac{11\pi/12}{\pi} = \frac{x}{180} \Rightarrow x = \frac{11}{12}(180) = 165°$ d) $\frac{-0.9825}{\pi} = \frac{x}{180} \Rightarrow x = \frac{-0.9825(180)}{\pi}$

$= -56.3°$

3. $\pi - \frac{5\pi}{12} = \frac{7\pi}{12}$

4. $180° - 130° = 50°$

$\frac{50}{180} = \frac{x}{\pi} \Rightarrow x = \frac{50\pi}{180} = \frac{5\pi}{18}$

5. Third Angle: $\pi - 2\left(\frac{2\pi}{7}\right) = \frac{3\pi}{7}$

Complement: $\frac{\pi}{2} - \frac{3\pi}{7} = \frac{7\pi}{14} - \frac{6\pi}{14} = \frac{\pi}{14}$

6. Convert 24° to radians:

$\frac{24}{180} = \frac{y}{\pi} \Rightarrow y = \frac{2\pi}{15}$

$\therefore \frac{\pi}{x} = \frac{2\pi}{15} \Rightarrow x = \frac{15}{2}$ or 7.5

7. $\frac{2\pi}{15} + \frac{\pi}{x} = \pi$

$\frac{2}{15} + \frac{1}{x} = 1$

$2x + 15 = 15x$

$15 = 13x$

$x = 15/13$

8. a) Quadrant IV $\Rightarrow 3\pi/2 < \theta < 2\pi$ or $(3\pi/2, 2\pi)$

b) Quadrant II $\Rightarrow \pi/2 \leq \theta < \pi$ or $[\pi/2, \pi)$

(Note: $\cos\theta = 0$ is included in the solution)

c) Quadrant I and III $\Rightarrow 0 < \theta < \pi/2 \cup \pi < \theta < 3\pi/2$

or $(0, \pi/2) \cup (\pi, 3\pi/2)$

d) Quadrant II and III $\Rightarrow \pi/2 \leq \theta \leq 3\pi/2$ or $[\pi/2, 3\pi/2]$

(Note: $\cos\theta = 0$ is included in the solution)

Exercise 1: Degree and Radian Measure (continued)

9. a) $\underline{III}, \underline{IV}$ b) $\underline{IV}$

10. a) $\underline{II}, \underline{III}$ b) $\underline{III}$

11. $\theta = 48^\circ = \frac{48\pi}{180}$ radians

$r = 24$ inches

$S = \theta r$

$s = \left(\frac{48\pi}{180}\right)(24) = \frac{32\pi}{5}$

12. $d = 16.4 \Rightarrow r = 8.2$

$S = \theta r$

$12.3 = \theta(8.2)$

$\theta = \frac{12.3}{8.2} = 1.5$

$\frac{1.5}{\pi} = \frac{x}{180} \Rightarrow x = \frac{1.5(180)}{\pi} = 85.9^\circ$

13. $6\sqrt{12} + 2\sqrt{27}$

$= 6(2\sqrt{3}) + 2(3\sqrt{3})$

$= 12\sqrt{3} + 6\sqrt{3}$

$= 18\sqrt{3}$

14. $x = 2$ gives $3^0 = 5^0 = 1$

No other answers are possible since, a part from 1, the powers of 3 and the powers of 5 are distinct.

15. $x^{1/3 + 1/6} = x^{1/2}$

16. a) $(x+3)^2 + 2 = 20$

$x^2 + 6x + 9 + 2 = 20$

$x^2 + 6x - 9 = 0$

$x = \frac{-6 \pm \sqrt{36 - 4(1)(-9)}}{2}$

$x = \frac{-6 \pm \sqrt{72}}{2} = \frac{-6 \pm 6\sqrt{2}}{2}$

$x = -3 \pm 3\sqrt{2}$

b) $\frac{6}{x} - \frac{x-1}{2} = 4$

$12 - (x^2 - x) = 8x$

$0 = x^2 + 7x - 12$

$x = \frac{-7 \pm \sqrt{49 - 4(1)(-12)}}{2}$

$= \frac{-7 \pm \sqrt{97}}{2}$

17. $y = mx + b$

$y = 2/3x + 12$

Let $y = 0 \Rightarrow 0 = \frac{2}{3}x + 12$

$-12 = 2/3x$

$-36 = 2x$

$-18 = x$

$\therefore$ the x-intercept is -18

18. a) $x(x^2 + 2x - 3) = x(x+3)(x-1)$

b) $x = 0$, $x = -3$, $x = 1$

19. $y = 6 - x$

$y = 1/2x^2 + x$

$\Rightarrow 6 - x = 1/2x^2 + x$

$12 - 2x = x^2 + 2x$

$0 = x^2 + 4x - 12$

$0 = (x+6)(x-2)$

$x = -6$ or $x = 2$

$\therefore$ The points are $(-6, 12)$ and $(2, 4)$

20. $ax + b = cx + d$

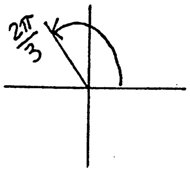
$ax - cx = d - b$

$x(a - c) = d - b$

$x = \frac{d-b}{a-c}$

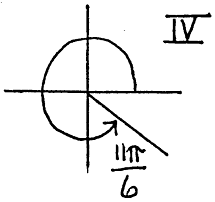
Exercise 2: The Unit Circle

1\. a)



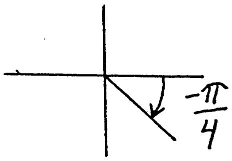
II

b)



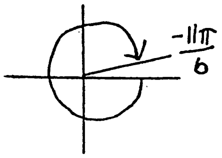
IV

c)



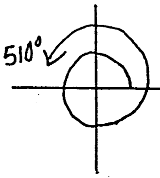
IV

d)



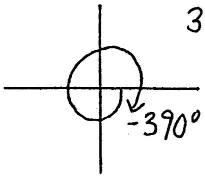
I

2\. a)



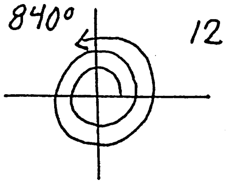
150°

b)



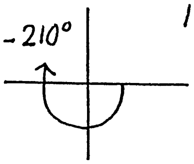
330°

c)



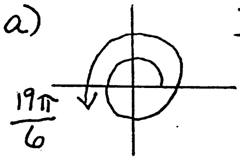
120°

d)



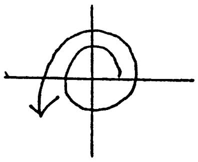
150°

3\. a)



$\frac{7\pi}{6}$

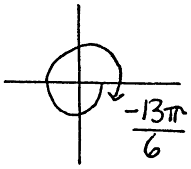
b)



$\frac{125\pi}{4} = 31\pi + \frac{\pi}{4}$

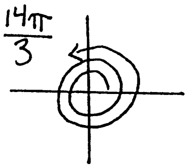
$\therefore$ coterminal angle $\frac{5\pi}{4}$

c)



$\frac{11\pi}{6}$

d)



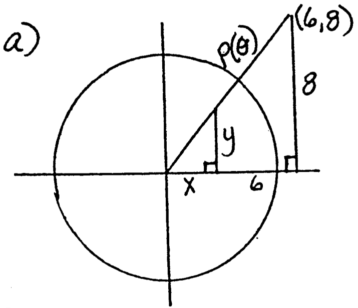
$\frac{14\pi}{3} = 4\pi + \frac{2\pi}{3}$

$\therefore$ coterminal angle $\frac{2\pi}{3}$

4\. a) I, II b) II, IV c) II d) IV

Exercise 2: The Unit Circle (continued)

5\. a)



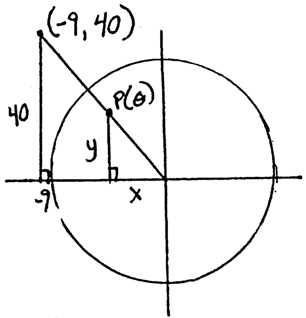
$\frac{y}{8}=\frac{1}{10} \Rightarrow y=\frac{4}{5}$

$\frac{x}{6}=\frac{1}{10} \Rightarrow x=\frac{3}{5}$

$6^2+8^2=r^2 \Rightarrow r=10$

$\therefore P(\theta)=(3/5, 4/5)$

b)



$\therefore \frac{y}{40}=\frac{1}{41} \Rightarrow y=\frac{40}{41}$

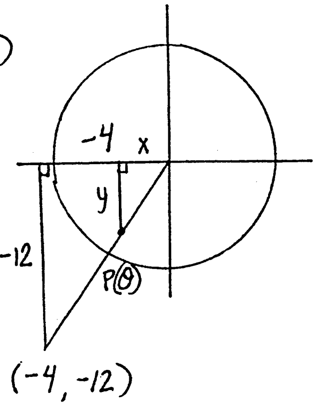
$\therefore \frac{x}{-9}=\frac{1}{41} \Rightarrow x=\frac{-9}{41}$

$\therefore P(\theta)=(-9/41, 40/41)$

$(-9)^2+(40)^2=r^2$

$r=41$

c)



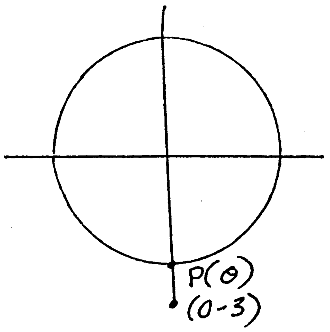
$\therefore \frac{y}{-12}=\frac{1}{4\sqrt{10}} \Rightarrow y=\frac{-3}{\sqrt{10}}=\frac{-3\sqrt{10}}{10}$

$\frac{x}{-4}=\frac{1}{4\sqrt{10}} \Rightarrow x=\frac{-1}{\sqrt{10}}=\frac{-\sqrt{10}}{10}$

$(-4)^2+(-12)^2=r^2$

$\Rightarrow r=\sqrt{160}$ or $4\sqrt{10}$

d) P(θ) is on the unit circle ⇒ $P(\theta)=(0,-1)$



Exercise 2: The Unit Circle (continued)

6. $x^2+y^2=1$

$(5/13)^2+y^2=1$

$y^2=1-25/169$

$y^2=\frac{144}{169}$

$y=\pm\frac{12}{13}$

since the point is in Quad IV

$y=\frac{-12}{13}$

7. $x^2+y^2=1$

$x^2+(8/17)^2=1$

$x^2=1-64/289$

$x^2=\frac{225}{289}$

$x=\pm\frac{15}{17}$

since the point is in Quad II

$\cos\theta=\frac{-15}{17}$

8. $\tan\theta=\frac{\sin\theta}{\cos\theta}$

$\tan\theta=\frac{-\sqrt{10}/4}{\sqrt{6}/4}$

$=\frac{-\sqrt{10}}{\sqrt{6}}=\frac{-\sqrt{60}}{6}$

$=\frac{-2\sqrt{15}}{6}=\frac{-\sqrt{15}}{3}$

9. $x^2+y^2=1$ – Equation of unit circle

$\left(\frac{\sqrt{5}}{5}\right)^2+\left(\frac{2\sqrt{5}}{5}\right)^2=\frac{5}{25}+\frac{20}{25}$

$=\frac{25}{25}=1$

$\therefore$ Yes $\left(\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right)$ is on the unit circle

10. $C=2\pi r$

$30=2\pi r\Rightarrow r=15/\pi$

$s=\theta r$

$5=\theta\left(\frac{15}{\pi}\right)$

$\theta=\frac{\pi}{3}$ radians $=60°$

11. since $s=\theta r$ and $A=\pi r^2$,

$A=\frac{\pi s^2}{\theta^2}$

$\theta=30°$ or $\pi/6$ radians

$A=\frac{\pi(7.6)^2}{(\pi/6)^2}=661.9\text{cm}^2$ (area of circle)

The slice would be 1/12th $[30°/360°]$ of the pie.

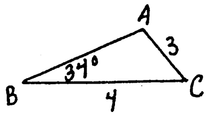
$\therefore$ Area of the slice $=\frac{661.9}{12}=55.2\text{cm}^2$

12. Students could explain:

- division of the unit circle
- how to determine if greater than one revolution
- values of each quadrantal angle
- approximate and/or exact values.

Exercise 2: The Unit Circle (continued)

13. solution 1: Using Law of sines



$$\frac{\sin A}{4} = \frac{\sin 34}{3} \Rightarrow \sin A = \frac{4\sin 34}{3} = 0.74559$$

$$\angle A = 48.21° \text{ or } 131.79°$$

Δ_1

$$\angle C = 180-34-48.21$$
$$= 97.79$$
$$\frac{c}{\sin 97.79} = \frac{3}{\sin 34}$$
$$c = \frac{3\sin 97.79}{\sin 34} = 5.315$$

Δ_2

$$\angle C = 180-34-131.79° = 14.21°$$
$$\frac{c}{\sin 14.21} = \frac{3}{\sin 34}$$
$$c = \frac{3\sin 14.21}{\sin 34} = 1.317$$

solution 2: Using Law of cosines

$$3^2 = 4^2 + x^2 - 2(4)(x)\cos 34° \Rightarrow 0 = x^2 - 6.6323x + 7$$

By the Quadratic Formula: $x = \frac{6.6323 \pm \sqrt{(6.6323)^2 - 28}}{2}$

$$x = 5.315 \text{ or } 1.317$$

14. $\sqrt{x^2+7} = 4$
$$x^2+7 = 16$$
$$x^2-9 = 0$$
$$(x-3)(x+3) = 0$$
$$x = 3 \text{ or } x = -3$$
check: $x = 3 \Rightarrow \sqrt{3^2+7} = \sqrt{16} = 4$
$x = -3 \Rightarrow \sqrt{(-3)^2+7} = \sqrt{16} = 4$

16. $y = (x^2+6x+9)+10-9$
$$y = (x+3)^2+1$$
$\therefore$ vertex A is $(-3,1)$
$$OA = \sqrt{(-3)^2+(1)^2} = \sqrt{10}$$

15. $2(k+2)+3 = k+2k+3$
$$2k+7 = 3k+3$$
$$4 = k$$

17. Point A: $x=2 \Rightarrow y = \frac{6}{2} = 3 \therefore A(2,3)$

Point B: $x=6 \Rightarrow y = \frac{6}{6} = 1 \therefore B(6,1)$

a) Area $= \frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$
$$A = \frac{1}{2}(3+1)(4) = 8$$

b) Perimeter = AB+BC+CD+DA
$$P = \sqrt{(2-6)^2+(3-1)^2}+1+4+3$$
$$P = 2\sqrt{5}+8$$

Exercise 2: The Unit Circle (continued)

18. $x^4 - 16 = (x^2 - 4)(x^2 + 4)$

$= (x - 2)(x + 2)(x^2 + 4)$

19. $12x^2 - 25x + 12 = 0$

$(3x - 4)(4x - 3) = 0$

$x = 4/3$ or $x = 3/4$

20. $y = (10x^2 - 9x \quad) + 2$

$y = 10\left(x^2 - \frac{9}{10}x + \left(\frac{9}{20}\right)^2\right) + 2 - \frac{810}{400}$

$y = 10\left(x - \frac{9}{20}\right)^2 - \frac{1}{40}$

Exercise 3: Special Angles and the Trigonometric Functions

1. Values read from the unit circle.

a) $1/2$ b) $-\frac{\sqrt{3}}{2}$ c) -1

d) $\frac{1}{\frac{1}{2}}$ or 2 e) $\frac{\frac{-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$, f) $\frac{\frac{-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

2. Values read from the unit circle:

a) $\frac{\sqrt{2}}{2}$ b) $-\frac{\sqrt{3}}{2}$

c) $\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}$ or $-\frac{\sqrt{3}}{3}$ d) $\frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$ or $\sqrt{2}$

3. a) $(1)(1)\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$ b) $(-\sqrt{3})\left(\frac{-1}{\sqrt{2}}\right) + (-1)\left(\frac{-1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{3}}$

$= \frac{3+\sqrt{2}}{\sqrt{6}} = \frac{3\sqrt{6}+2\sqrt{3}}{6}$

4. a) LHS: $\frac{\sin \pi/3}{1+\cos \pi/3} = \frac{\sqrt{3}/2}{1+1/2} = \frac{\sqrt{3}/2}{3/2} = \frac{\sqrt{3}}{3}$

RHS: $\frac{1-\cos \pi/3}{\sin \pi/3} = \frac{1-1/2}{\sqrt{3}/2} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

b) LHS: $\sin \pi/6 = 1/2$

RHS: $\sqrt{\frac{1-\cos \pi/3}{2}} = \sqrt{\frac{1-1/2}{2}}$

$= \sqrt{\frac{1/2}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

5. If $\tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \sin \theta = \pm\frac{1}{2}$ and $\cos \theta = \pm\frac{\sqrt{3}}{2}$ (both positive or both negative)

a) In Quadrant I $\Rightarrow \theta = \pi/6$ b) In Quadrant II and III $\Rightarrow \theta = 7\pi/6$

c) In Quadrant IV $\Rightarrow$ not possible ($\sin \theta$ is negative and $\cos \theta$ is positive)

6. a) In Quadrant III or IV

$\Rightarrow \theta = \frac{5\pi}{4}; \frac{7\pi}{4}$

b) In Quadrant I or IV

$\theta = \frac{\pi}{6}; \frac{11\pi}{6}$

c) In Quadrant III

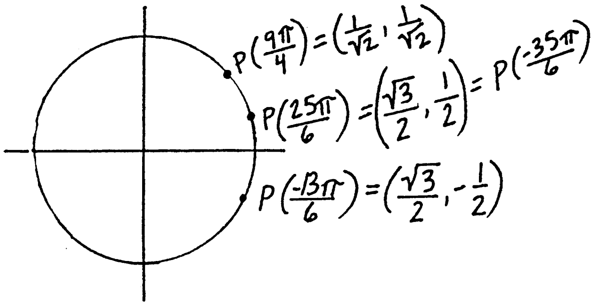
$\theta = \frac{4\pi}{3}$

d) In Quadrant II

$\theta = \frac{2\pi}{3}$

Exercise 3: Special Angles and the Trigonometric Functions (continued)

7.



a) $\frac{\sqrt{3}}{2}$ b) $\frac{\sqrt{2}}{2}$

c) $\frac{-\sqrt{3}}{3}$ d) 2

8. $(1)\times 2: \ 8x - 6y = 20$

$(2)\times 3: \ 9x + 6y = -3$

$17x = 17$

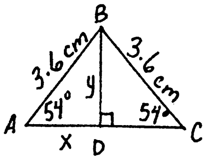
$x = 1$

$4(1) - 3y = 10$ $\therefore (x, y) = (1, -2)$

$-3y = 6$

$y = -2$

9.



$\sin 54° = \frac{y}{36}$

$y = 2.91246$

$\cos 54° = \frac{x}{3.6}$

$x = 2.11603 \Rightarrow$

$AC = 4.23206$

a) Perimeter: $3.6 + 3.6 + 4.23206 \approx 11.4$cm

b) Area: $\frac{1}{2}(4.23206)(2.91246) \approx 6.2\text{cm}^2$

10. Quadrant IV $\Rightarrow \frac{3\pi}{2} < \theta < 2\pi$

or $\left(\frac{3\pi}{2}, 2\pi\right)$

11. $\frac{\frac{11\pi}{24}}{\pi} = \frac{x}{180} \rightarrow x = \frac{11}{24}(180) = 82.5°$

12. Right Isosceles $\Delta \Rightarrow 90°, 45°, 45°$

$90° = \pi/2$; $45° = \pi/4$

$\therefore \pi/2, \pi/4, \pi/4$ radians

13. $\frac{\frac{5\pi}{12}}{\pi} = \frac{x}{180} \rightarrow x = \frac{5}{2}(180) = 75°$

Since corresponding angles are equal, the lines are parallel.

14. $x^2 + y^2 = 1$

$x^2 + (1/3)^2 = 1$

$x^2 = 1 - 1/9$

$x^2 = 8/9$

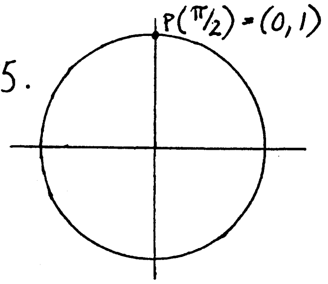
$x = \pm \frac{2\sqrt{2}}{3}$

Since $\sin\theta > 0$ and $\cos\theta < 0 \Rightarrow P(\theta)$ is in Quad II.

$\therefore x = -2\sqrt{2}/3$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/3}{-2\sqrt{2}/3} = \frac{1}{-2\sqrt{2}} = \frac{-\sqrt{2}}{4}$

15.



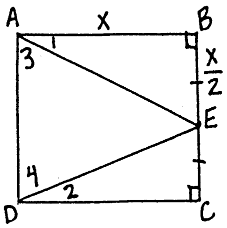
$P(\theta)$ always becomes (0, 1)

$\therefore \sin\theta = 1$, $\cos\theta = 0$, and $\tan\theta$ is undefined

Exercise 3: Special Angles and the Trigonometric Functions (continued)

16. $(1) - \left(\frac{1}{2}\right) = \frac{1}{2}$

17.



Let $AB = x$ units

$\Rightarrow BE = \frac{x}{2}$ units

$\therefore \tan\angle 1 = {}^{x/2}/_{x} = \frac{1}{2}$

$\angle 1 = 26.57°$

Similarily $\angle 2 = 26.57°$

$\angle 3 = 90 - 26.57 = 63.43°$

$\angle 4 = \angle 3 = 63.43°$

$\therefore \angle AED = 180° - 63.43 - 63.43 = 53.14$

or $53°$

18. $\sqrt{2x+3} = 1 + \sqrt{x+1}$

$2x+3 = 1 + 2\sqrt{x+1} + x + 1$

$x+1 = 2\sqrt{x+1}$

$x^2 + 2x + 1 = 4(x+1)$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

Check:

$x = 3 \quad \sqrt{2(3)+3} = \sqrt{9} = 3$

$1 + \sqrt{3+1} = 1 + \sqrt{4} = 3$

$x = -1 \quad \sqrt{2(-1)+3} = \sqrt{1} = 1$

$1 + \sqrt{-1+1} = 1 + 0 =$

$\therefore x = 3$ or $x = -1$

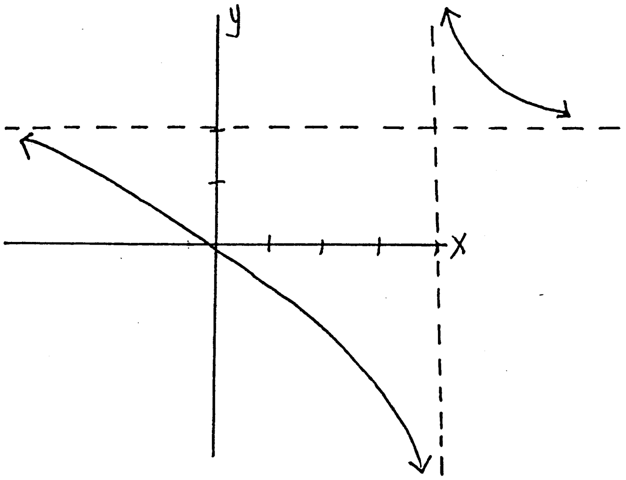
19. $(2, 0)$ is a point on $3x + 4y = 6$

The perpendicular distance from $(2, 0)$ to $3x + 4y = 1$ is:

$$d = \frac{|3(2) + 4(0) - 1|}{\sqrt{3^2 + 4^2}} = \frac{5}{5} = 1$$

20. $H(x) = \dfrac{2x+1}{x-4}$

x-intercept:	y-intercept:
$\Rightarrow y = 0$	$\Rightarrow x = 0$
$0 = \dfrac{2x+1}{x-4}$	$y = \dfrac{2(0)+1}{0-4}$
$0 = 2x+1$	$y = \dfrac{-1}{4}$
$x = -\dfrac{1}{2}$	



Exercise 4: Solving Trigonometric Equations on a Specified Interval

1. a) $2\cos\theta = 2$

$\cos\theta = 1$

$\theta = 0°, 360°$

b) $5\tan\theta + 4 = 0$

$\tan\theta = {}^-4/5$

$\theta = 38.7°$ (related angle)

$\therefore \theta = 141.3°$ or $321.3°$

c) $4\tan\theta - 7 = 5\tan\theta - 6$

$-1 = \tan\theta$

$\theta = 135°$ or $315°$

d) $2\sin^2\theta + \sin\theta = 0$

$\sin\theta\,(2\sin\theta + 1) = 0$

$\sin\theta = 0$ or $2\sin\theta + 1 = 0$

$\theta = 0°, 180°, 360°$ $\sin\theta = {}^-1/2$

$\theta = 210°, 330°$

$\therefore \theta = 0°, 180°, 210°, 330°, 360°$

2. $2\sin^2\theta - \sin\theta = 0$

$\sin\theta\,(2\sin\theta - 1) = 0$

$\sin\theta = 0$ or $2\sin\theta - 1 = 0$

$\sin\theta = 1/2$

$\theta = 0, \pi, 2\pi$ $\theta = \pi/6, 5\pi/6$

3. $\tan\theta + \sqrt{3} = 0$

$\tan\theta = -\sqrt{3}$

$\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$

4. $2\tan\theta + 2\sqrt{3} = 0$

$2\tan\theta = -2\sqrt{3}$

$\tan\theta = -\sqrt{3}$

$\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$

5. $2\cos\theta + \sqrt{3} = 0$

$2\cos\theta = -\sqrt{3}$

$\cos\theta = -\sqrt{3}/2$

$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$

6. $4\cos^2\theta = 1$

$4\cos^2\theta - 1 = 0$

$(2\cos\theta - 1)(2\cos\theta + 1) = 0$

$\cos\theta = 1/2$ or $\cos\theta = {}^-1/2$

$\theta = 2\pi/3, 4\pi/3$

7. $2\sin\theta + \sqrt{2} = 0$

$2\sin\theta = -\sqrt{2}$

$\sin\theta = {}^-\sqrt{2}/2$

$\theta = \frac{5\pi}{4}$

8. $2\cos^2\theta - 5\cos\theta - 3 = 0$

$(2\cos\theta + 1)(\cos\theta - 3) = 0$

$2\cos\theta + 1 = 0$ or $\cos\theta - 3 = 0$

$\cos\theta = {}^-1/2$ $\cos\theta = 3$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ no solution

9. $\left(\frac{\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right)(1) = \frac{-3}{4}$

10. LHS: $2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 = \frac{3}{2} - 1 = 1/2$

RHS: $\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = 1/2$

11. $\theta = \frac{4\pi}{3}$

12. $x^2 + y^2 = 1$

$({}^-3/4)^2 + y^2 = 1$

$y^2 = 1 - 9/16$

$y^2 = 7/16 \rightarrow y = \pm\frac{\sqrt{7}}{4}$

$\therefore \sin\theta = \frac{^-\sqrt{7}}{4}$

$\cos\theta < 0$ and $\tan\theta > 0$

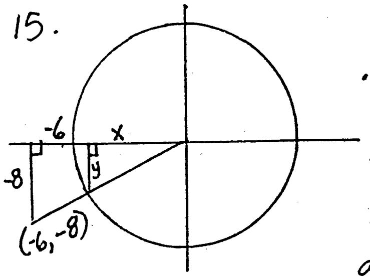
$\therefore P(\theta)$ in III

Exercise 4: Solving Trigonometric Equations on a Specified Interval (continued)

13. Method 1: Students could recognize that 27π is an odd multiple of π thus falling on the quadrantal angle of π. Therefore it is not located in a quadrant and its coordinates are $(-1, 0)$.
Method 2: Students could determine the number of revolutions the angle wraps by dividing 27π by 2π. $13\frac{1}{2}$ revolutions would take you to the quadrantal angle of π. see final answer from method 1.

14. (c) and (e) are incorrect because the interval includes the angle $\pi/4$, which is where $\cos\theta = \sin\theta$
(b) is incorrect because in the interval $\pi/4 < \theta \leq \pi/2$ $\sin\theta > \cos\theta$
(a) is incorrect because that interval includes $\cos\theta > \sin\theta$, $\cos\theta = \sin\theta$ and $\sin\theta > \cos\theta$.
(d) is correct since all angles in this interval have a greater cosine value than sine.

15.



$(-6)^2 + (-8)^2 = r^2 \Rightarrow r = 10$

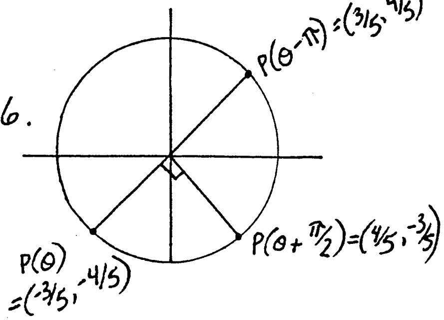
$\therefore \frac{y}{-8} = \frac{1}{10} \Rightarrow y = -\frac{4}{5}$

$\frac{x}{-6} = \frac{1}{10} \Rightarrow x = \frac{-3}{5}$

$\therefore P(\theta) = (-3/5, -4/5)$

and $\cos\theta = -3/5$

16.



By congruent △'s, $P(\theta + \pi/2) = (4/5, -3/5)$ $P(\theta - \pi) = (-3/5, 4/5)$

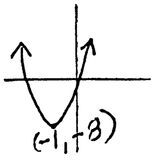
a) $\cos(\theta + \pi/2) = 4/5$

b) $\cos(\theta - \pi) = 3/5$

17. $(1)(-1) = -1$

18. The parabola opens up so $a > 0$
There are no x-intercepts so
$b^2 - 4ac < 0$ ∴ B

19. $f(x) = 3(x^2 + 2x + 1) - 5 - 3$
$f(x) = 3(x+1)^2 - 8$
$v(-1, -8)$
Domain: $\{x \in \mathbb{R}\}$
Range: $\{y \mid y \geq -8\}$



20. The slope from $(-1, 1)$ to $(0, 2)$ is
$\frac{2-1}{0+1} = 1$
∴ Line AB has equation $y = x + 2$

Solve: $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$ $\Rightarrow x^2 = x + 2$
$x^2 - x - 2 = 0$
$(x-2)(x+1) = 0$
$x = 2$ or $x = -1$ ∴ B is the point $(2, 4)$

Exercise 5: General Solution of Trigonometric Equations

1\. $\sin\theta = 1/2$

$\theta = \pi/6$ -related angle

$\theta = \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi,$

where k is an integer.

2\. a) $\cos 3\theta = 1/2$

$3\theta = \pi/3$ -related angle

$3\theta = \pi/3 + 2k\pi, \frac{5\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{9} + \frac{2k\pi}{3}, \frac{5\pi}{9} + \frac{2k\pi}{3}$ where k is an integer.

2\. b) $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2}$

$\frac{\theta}{2} = \frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi$

$\theta = \frac{2\pi}{3} + 4k\pi, \frac{4\pi}{3} + 4k\pi$

where k is an integer.

3\. $\tan\theta = 0$ $\theta = k\pi$ where k is an integer

4\. $\sin^2\theta = 1/4$

$\sin\theta = \pm 1/2$

$\theta = \pi/6$ -related angle

$\theta = \pi/6 + 2k\pi, 5\pi/6 + 2k\pi, 7\pi/6 + 2k\pi$ $11\pi/6 + 2k\pi$

where k is an integer.

5\. $(1+\sin\theta)(1-\cos\theta) = 0$

$\sin\theta = -1$ or $\cos\theta = 1$

$\theta = 3\pi/2 + 2k\pi$ $\theta = 0 + 2k\pi$

$\theta = 3\pi/2 + 2k\pi, 2k\pi$ where k is an integer.

6\. $2\sec\theta + 4 = 0$

$\sec\theta = -2$

$\cos\theta = -1/2$

$\theta = \pi/3$ -related angle

$\theta = 2\pi/3 + 2k\pi, 4\pi/3 + 2k\pi$

where k is an integer.

7\. $4\csc\theta + 6 = 14$

$4\csc\theta = 8$

$\csc = 2$

$\sin\theta = 1/2$

$\theta = \pi/6$ -related angle

$\theta = \pi/6 + 2k\pi, 5\pi/6 + 2k\pi,$

where k is an integer.

8\. $(\sin\theta - 1)(2\sec\theta + 1) = 0$

$\sin\theta = 1$ $\sec\theta = -1/2$

$\theta = \pi/2 + 2k\pi$ $\cos\theta = -2$

no solution

$\therefore \theta = \pi/2 + 2k\pi$, where k is an integer.

9\. $4\sin^2\theta - 3 = 0$

$4\sin^2\theta = 3$

$\sin^2\theta = 3/4$

$\sin\theta = \pm\sqrt{3}/2$

$\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$

$-\pi/3, -2\pi/3, -4\pi/3, -5\pi/3$

Exercise 5: General Solution of Trigonometric Equations (continued)

10. $\{-2\pi \le 0 \le 2\pi\}$

$5\tan\theta + 5 = 0$

$5\tan\theta = -5$

$\tan\theta = -1$

$\theta = 3\pi/4,\ 7\pi/4,\ -\pi/4,\ -5\pi/4$

11. $\cos\theta + \cos^2\theta + \sin^2\theta = 0$

$\cos\theta + 1 = 0$

$\cos\theta = -1$

$\theta = \pi$

12. $2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = 1/2$ or $\sin\theta = -1$

$\theta = \pi/6,\ 5\pi/6 \quad \theta = 3\pi/2$

$\therefore \theta = \pi/6,\ 5\pi/6,\ 3\pi/2$

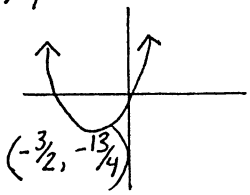
13. $f(x) = x^2 + 3x - 1$

$f(x) = (x^2 + 3x + (\frac{3}{2})^2) - 1 - \frac{9}{4}$

$f(x) = (x + 3/2)^2 - 13/4$

$\therefore$ Range:

$\{y \mid y \ge -\frac{13}{4}\}$ or $[-\frac{13}{4}, \infty)$



14. $80\text{cm} = 0.8\text{M} = d$

$\therefore r = \frac{0.8}{2} = 0.4\text{M}$

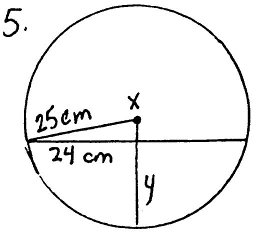
$S = \theta r$

$130 = \theta(0.4)$

$\theta = 325$ radians

revolutions : $\frac{325}{2\pi} = 51.73$ rev.

15.

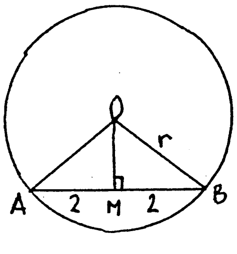


$x^2 + 24^2 = 25^2$

$x = 7$

$y = 25 - 7 = 18\text{cm}$

16.



a) Let O be the center of the circle and let AB be one side of the pentagon.

The measure of $\angle AOB = 1/5(360) = 72°$

Draw a $\perp$ to AB then $MB = 2$ and $\angle MOB = 36°$

$\sin 36° = \frac{2}{r} \Rightarrow r = \frac{2}{\sin 36°} = 3.40$

b) The area of $\triangle AOB = \frac{1}{2}(3.40)^2 \sin 72 = 5.506\ u^2$

$\therefore$ The area of the pentagon is $5(5.506) = 27.53\ u^2$

17. $A = 5000(1.06)^5 = \$6691.13$

18. The line is $2x - 3y + 18 = 0$

$d = \frac{|2(-1) - 3(3) + 18|}{\sqrt{4+9}} = \frac{7}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}}$

$= \frac{7\sqrt{13}}{13}$

Exercise 5: General Solution of Trigonometric Equations (continued)

19. $ax+c=b(x-c)$

$ax+c=bx-bc$

$ax-bx=-bc-c$

$x(a-b)=-bc-c$

$x=\dfrac{-bc-c}{a-b}$

20. $x+y=7$

$\underline{2x-y-2z=12}$

$3x-2z=19$

$(2)\ x-2 \quad \underline{-2x+2z=-16}$

$x=3$

check: $3+4=7$ ✓

$3+5=8$ ✓

$2(3)-4-2(-5)=12$ ✓

$3+y=7$

$y=4$

$3-z=8$

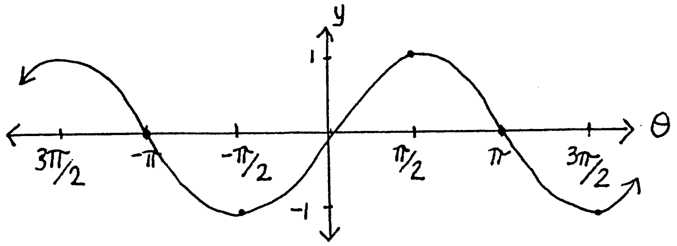
$-5=z$

Exercise 6: Graphing Circular Functions

1.

θ	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$
$\sin\theta$	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1

Note: $1/\sqrt{2} \approx 0.7$

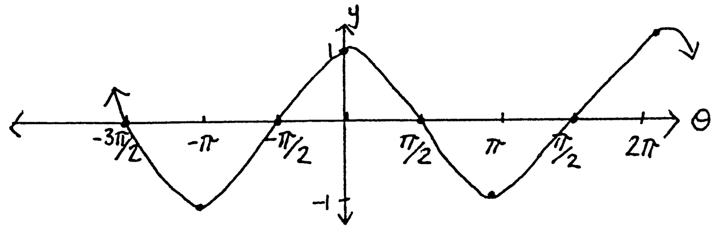


2. a) 1   b) 2π c) $\{x \in R\}$ d) $\{y \mid -1 \le y \le 1\}$ e) $k\pi$, where $k \in I$

3.

θ	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$
$\cos\theta$	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0

Note: $1/\sqrt{2} \approx 0.7$

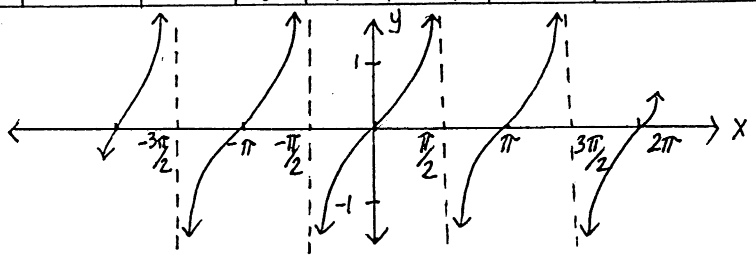


4. a) 1   b) 2π c) $\{x \in R\}$ d) $\{y \mid -1 \le y \le 1\}$ e) $\pi/2 + k\pi$, where $k \in I$

5. They are identical in amplitude, period, domain, and range. The curves would coincide if the cosine curve would undergo a horizontal shift of $\pi/2$ units to the right.

6.

x	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
$\tan x$	1	undef.	-1	0	1	undef.	-1	0	1	undef.	-1



Exercise 6: Graphing Circular Functions (continued)

7. a) undefined b) π c) All values except odd integral multiples of $\pi/2$, or $(2K+1)\frac{\pi}{2}$ where $K \in I$.

d) $\{y \in R\}$ e) $K\pi$, where $K \in I$

8. $x = (2K+1)\frac{\pi}{2}$ where $K \in I$

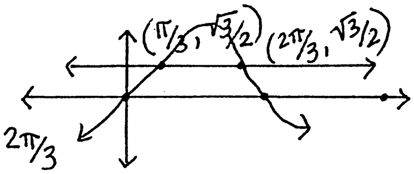
9. a) $(-2\pi, -3\pi/2) \cup (-\pi/2, \pi/2) \cup (3\pi/2, 2\pi)$
b) $(-2\pi, -\pi) \cup (0, \pi)$

10. $\sin^2\theta + \sin\theta - 4 = 0$

$$\sin\theta = \frac{-1 \pm \sqrt{1-(-16)}}{2}$$

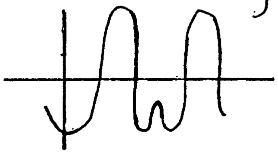
$\sin\theta = 1.561552813$
or
$\sin\theta = -2.561552813$
$\therefore$ No solution

11. $0 \le x \le 2\pi$ Graph $y = \sin x$
$\sin x = \sqrt{3}/2$ $y = \sqrt{3}/2$



$x = \pi/3, 2\pi/3$

12. Rough sketch from calculator
$y = 2\sin^2\theta + \sin\theta - 2$



calculate zeroes

$\theta = 0.896, 2.246$

13. $\sec\theta = \frac{\sqrt{7}}{2} \Rightarrow \cos\theta = \frac{2}{\sqrt{7}}$

$\sin^2\theta + \cos^2\theta = 1$
$\sin^2\theta + (2/\sqrt{7})^2 = 1$
$\sin^2\theta + 4/7 = 1$
$\sin^2\theta = 3/7$
$\sin\theta = \pm\sqrt{3/7}$
since θ is in IV, $\sin\theta = -\frac{\sqrt{3}}{\sqrt{7}}$
$\therefore \csc\theta = -\frac{\sqrt{7}}{\sqrt{3}} = -\frac{\sqrt{21}}{3}$

14. Radians:

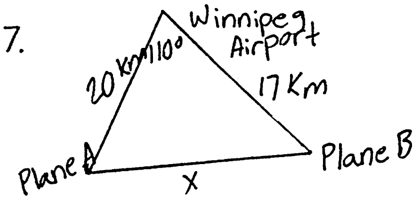
$$2\pi + \frac{240}{180}\pi = 2\pi + \frac{4\pi}{3} = \frac{10\pi}{3}$$

15. $\theta = 180° + 21.7° = 201.7°$
$\theta = 360° + 201.7° = 561.7°$

16. $P(B) = P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Exercise 6: Graphing Circular Functions (continued)

17.



$x^2 = 20^2 + 17^2 - 2(20)(17)\cos 110°$

$x^2 = 921.5736975$

$x = 30.36$ Km

18. Solution one:

$3m + b = 18$

$\underline{2m + b = 24}$

$m = -6$

$\therefore -18 + b = 18$

$b = 36$

$\therefore f(x) = -6x + 36$

Solution two:

The line passes through (3, 18) and (2, 24)

$M = \frac{24-18}{2-3} = -6$

$y = -6x + b$

$18 = -18 + b$

$36 = b$

$\therefore f(x) = -6x + 36$

19. $\sin 30° = \frac{y}{18}$

$y = 18(½)$

$y = 9$

$\cos 30° = \frac{x}{18}$

$x = 18\left(\frac{\sqrt{3}}{2}\right)$

$x = 9\sqrt{3}$

20. a) $f(3) = \frac{2(3)}{3-2} = 6$

$f(f(3)) = f(6) = \frac{2(6)}{6-2} = \frac{12}{4} = 3$

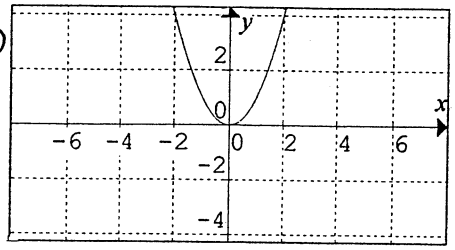
b) $f(x) = \frac{2x}{x-2}$

$f(f(x)) = f\left(\frac{2x}{x-2}\right)$

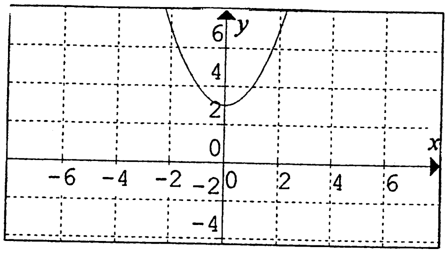
$$= \frac{2\left(\frac{2x}{x-2}\right)}{\frac{2x}{x-2} - 2} = \frac{\frac{4x}{x-2}}{\frac{2x-2x+4}{x-2}} = \frac{4x}{4} = x$$

Exercise 7: Translations

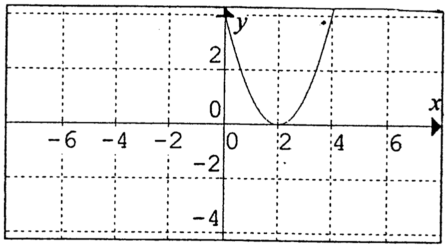
1. a)



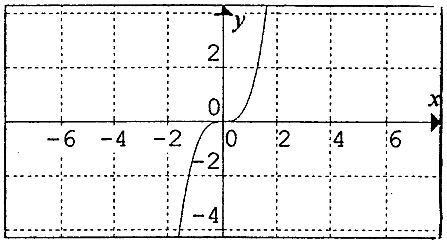
b)



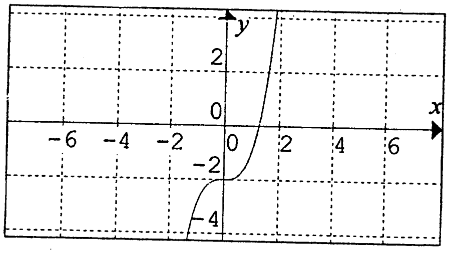
c)



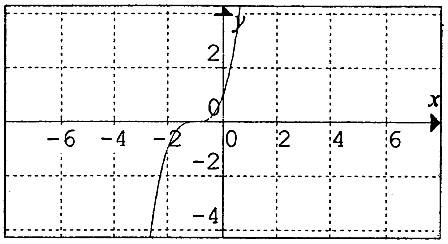
2. a)



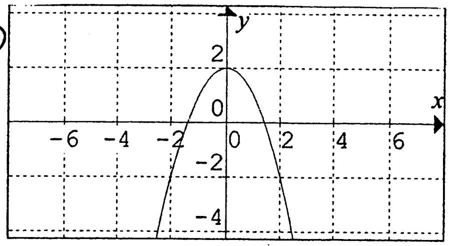
b)



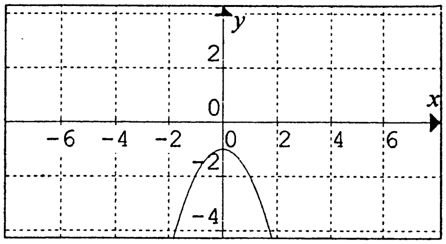
c)



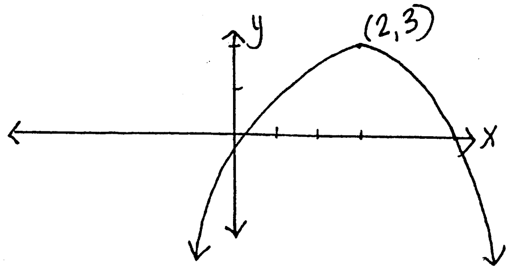
3. a)



b)

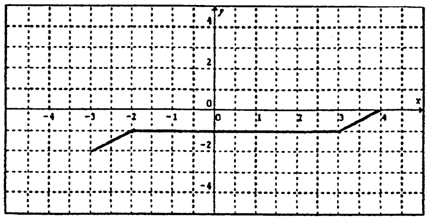


c)

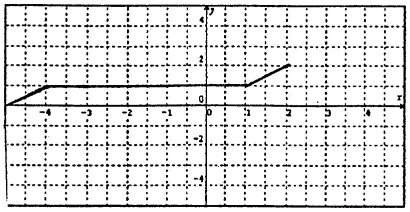


Exercise 7: Translations (continued)

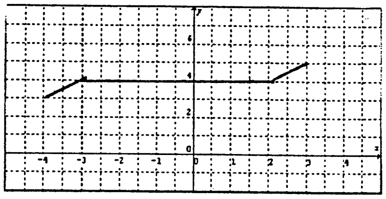
4. a)



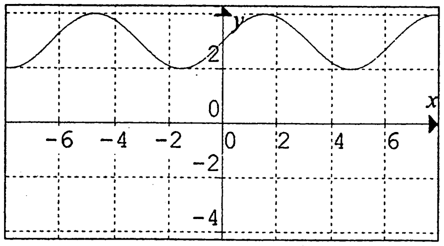
b)



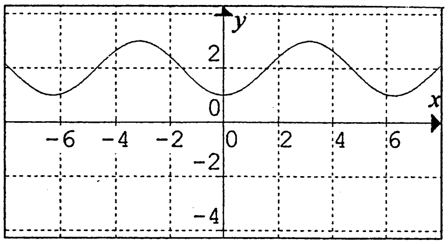
c)



5.

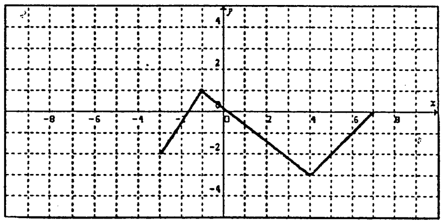


6.



7. $y = 3 + \sin x$	$y = \sin(x - \frac{\pi}{2}) + 2$
Range: $\{2 \le y \le 4\}$	Range: $\{1 \le y \le 3\}$
Period: 2π	Period: 2π
Amplitude: 1	Amplitude: 1

8.



9. a) $f(x) = g(x) + 4$

b) $g(x) = f(x) - 4$

10. $\sin 3\theta = -\frac{\sqrt{2}}{2}$

$3\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4}$

$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

11. $\tan\theta = \frac{6}{7}$

$\tan^2\theta + 1 = \sec^2\theta$

$(6/7)^2 + 1 = \sec^2\theta$

$\frac{36}{49} + 1 = \sec^2\theta$

$\frac{85}{49} = \sec^2\theta$

$\sec\theta = \pm\frac{\sqrt{85}}{7}$

Since θ is in III, $\sec\theta = \frac{-\sqrt{85}}{7}$

12. Domain: $\{x \in R\}$

Range: $\{y \mid 0 \le y \le 2\}$

Exercise 7: Translations (continued)

13. An inscribed angle is half the measure of its intercepted arc; therefore, $\angle PAC = \theta/2$

Using right Δ APB, $\tan \theta/2 = PB/AB = \frac{PB}{1+OB}$

In right Δ POB, $\sin\theta = \frac{PB}{1} = PB$

In right Δ POB, $\cos\theta = \frac{OB}{1} = OB$, $\therefore 1+\cos\theta = 1+OB = AB$

$\therefore \tan \theta/2 = \sin\theta \cancel{/}(1+\cos\theta)$

14. $\sin\theta = \pi$ $\sin\theta = \frac{\pi}{6}$

$(\sin\theta - 0)(\sin\theta - 1) = 0$

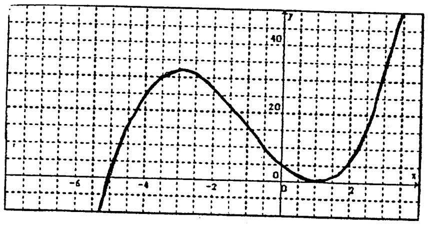
$\sin^2\theta - \sin\theta = 0$

or $\cos\theta = \pi$ $\cos\theta = \frac{\pi}{6}$

$(\cos\theta + 1)(\cos\theta - 0)$

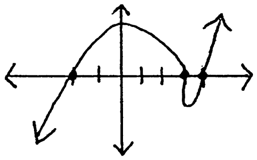
$\cos^2\theta + \cos\theta = 0$

15.



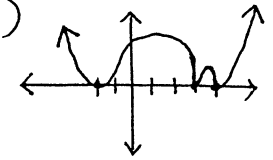
16. If the x-intercepts are -2, 3, 4 then the function could be $f(x) = a(x+2)^n(x-3)^M(x-4)^P$

i)



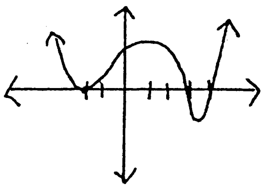
or

ii)



or

iii)



The above graphs show 3 ways that the function could have a y-intercept of 8. There are still more possibilities.

i) If $f(x) = a(x+2)(x-3)(x-4)$ then $8 = a(2)(-3)(-4)$

$\Rightarrow a = 1/3$

ii) If $f(x) = a(x+2)^2(x-3)^2(x-4)^2$ then $8 = a(2)^2(-3)^2(-4)^2$

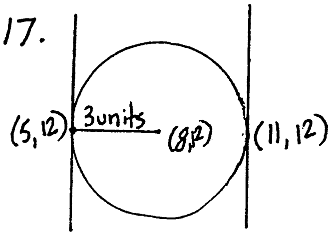
$\Rightarrow a = 1/72$

iii) If $f(x) = a(x+2)^2(x-3)(x-4)$ then $8 = a(2)^2(-3)(-4)$

$\Rightarrow a = 1/6$

Exercise 7: Translations (continued)

17.



$\therefore x = 5$ and $x = 11$

18. $f(75) = f(3 + 18 \cdot 4) = f(3) = 12$

19. a) $m = \frac{8}{4} = 2$, $b = 0$

$\therefore y = 2x$

b) $m = \frac{8-0}{4-8} = \frac{8}{-4} = -2$

$y - 8 = -2(x-4)$

$y - 8 = -2x + 8$

$y = -2x + 16$

20. $f(x) = (x+2)^2 - 3$

$y = (x+2+2)^2 - 3 + 4$

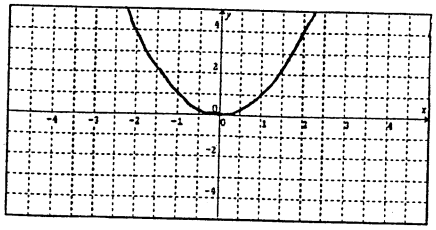
$y = (x+4)^2 + 1$

$y = x^2 + 8x + 16 + 1$

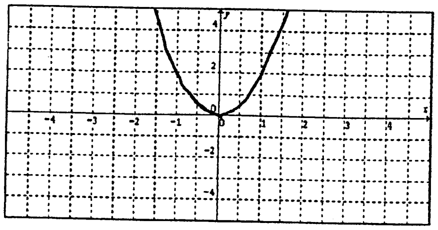
$y = x^2 + 8x + 17$

Exercise 8: Horizontal and Vertical Stretches

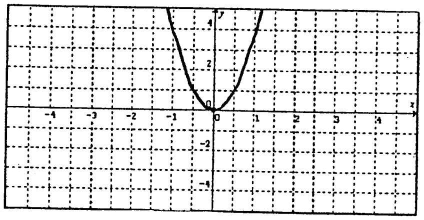
1. a)



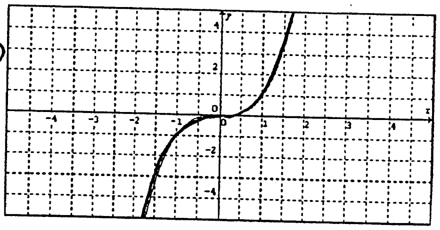
b)



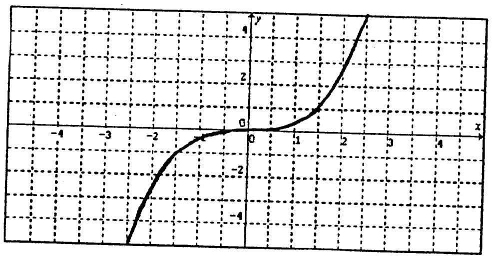
c)



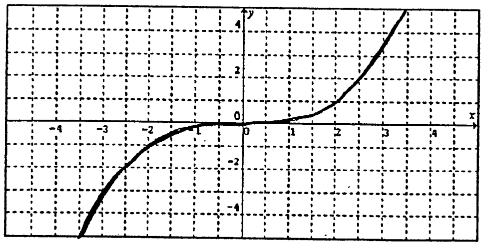
2. a)



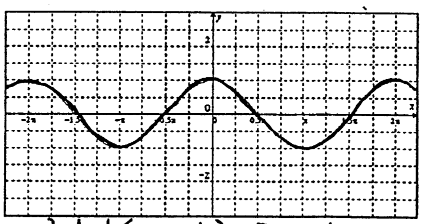
b)



c)



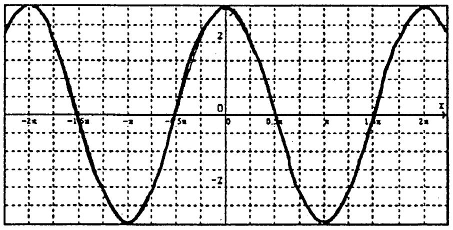
3. a)



Range: $\{y \mid -1 \le y \le 1\}$ Period: 2π

Amplitude: 1

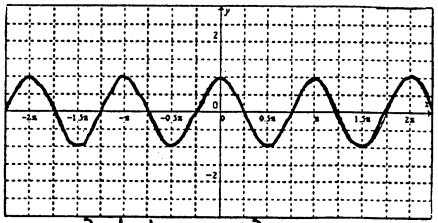
b)



Range: $\{y \mid -3 \le y \le 3\}$ Period: 2π

Amplitude: 3

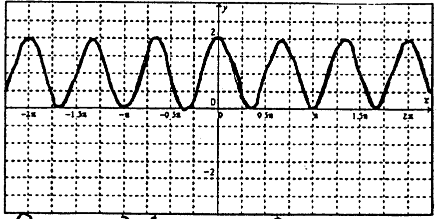
c)



Range: $\{y \mid -1 \le y \le 1\}$

Period: π Amplitude: 1

d)

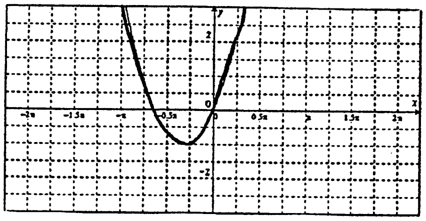


Range: $\{y \mid 0 \le y \le 2\}$

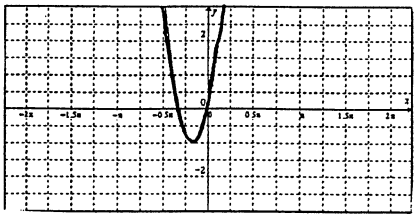
Period: $2\pi/3$ Amplitude: 1

Exercise 8: Horizontal and Vertical Stretches (continued)

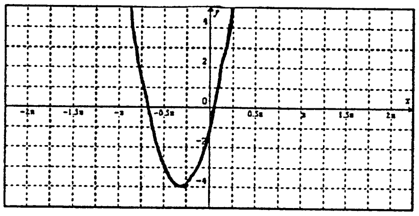
4. a)



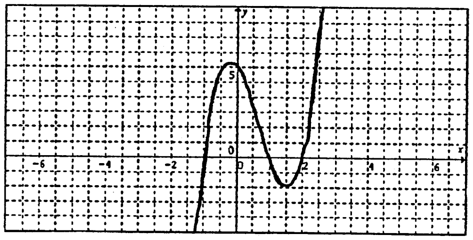
b)



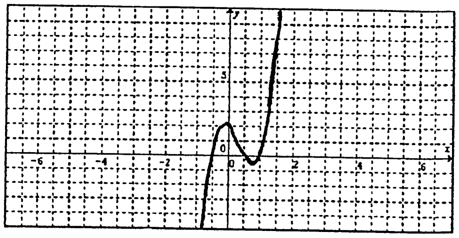
c)



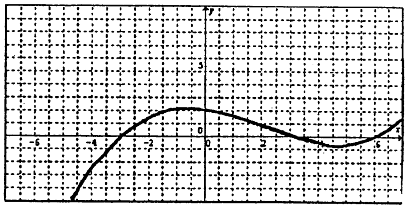
5. Vertical stretch by a factor of 3.



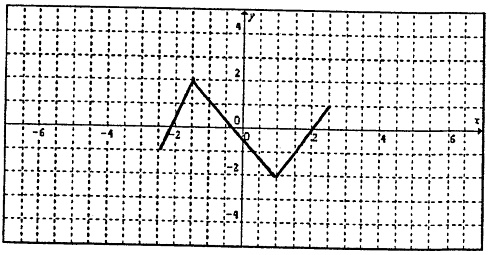
6. Horizontal stretch by a factor of ½



7. Horizontal stretch by a factor of 3



8.



9. a) $f(x) = g(2x)$
b) $g(x) = f(\frac{1}{2}x)$

10. $\frac{\sin^2 18° + \cos^2 18°}{1 - \cos^2 210°} = \frac{1}{1-(-\frac{\sqrt{3}}{2})^2} = \frac{1}{1-3/4}$

$= \frac{1}{1/4} = 4$

11. $(\sin\theta - 1)(\sin\theta - \frac{1}{2}) = 0$
$\sin^2\theta - \frac{3}{2}\sin\theta + \frac{1}{2} = 0$ or
$2\sin^2\theta - 3\sin\theta + 1 = 0$

$(\cos\theta - 0)(\cos\theta - \frac{\sqrt{3}}{2}) = 0$
$\cos^2\theta - \frac{\sqrt{3}}{2}\cos\theta = 0$
$2\cos^2\theta - \sqrt{3}\cos\theta = 0$

Exercise 8: Horizontal and Vertical Stretches (continued)

12. $\sin^2\theta + \cos^2\theta + 2\cos\theta = 0$

$1 + 2\cos\theta = 0$

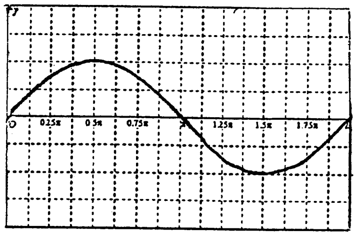
$2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$

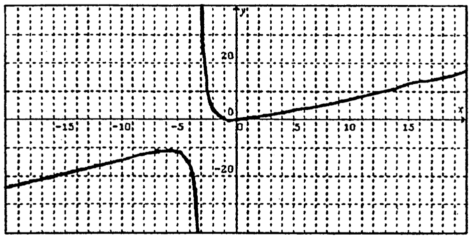
$\theta = \frac{2\pi}{3} + 2K\pi, \frac{4\pi}{3} + 2k\pi$

where $K \varepsilon I$

13.



14.

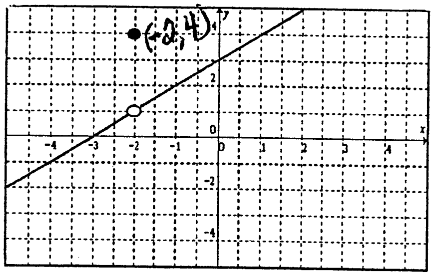


Domain $= \{x \mid x \neq -3\}$

Range $= \{y \mid y \leq -11.65 \text{ and } y \geq -0.34\}$

Note: values for range are approximate.

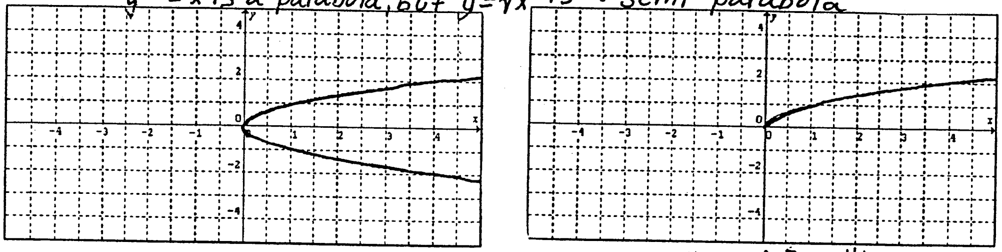
15.



16. since the central angle of a full circle measures 2π radians, $\theta/2 = 2\pi/2 = \pi$, and ∴ the area of the circle is $A = \pi r^2$

17.

$y^2 = x$ is a parabola, but $y = \sqrt{x}$ is a semi-parabola

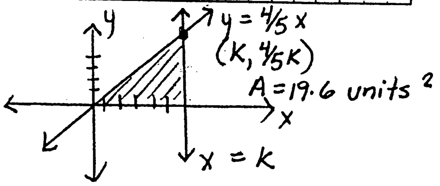


18. If $x = 3 \Rightarrow x^2 + 7x + K = 0$

$9 + 21 + K = 0$

$K = -30$

19.



$A = \frac{1}{2}(K)(\frac{4}{5}K) = 19.6$

$4K^2 = 196$

$K^2 = 49$

$K = \pm 7$

since $K > 0$, $K = 7$

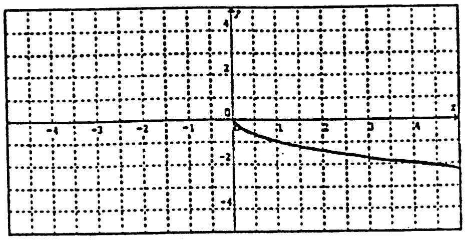
20. a) $A = \frac{\pi/4}{2\pi}(\pi(12)^2) = \frac{1}{8}(144\pi)$
$= 18\pi$

b) $A = \frac{\pi/3}{2\pi}(\pi r^2) = \frac{1}{6}\pi r^2$

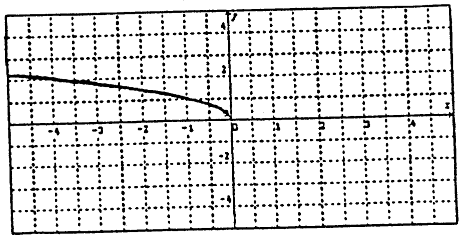
c) $A = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}\theta r^2$

Exercise 9: Symmetry, Reflections, and Inverses

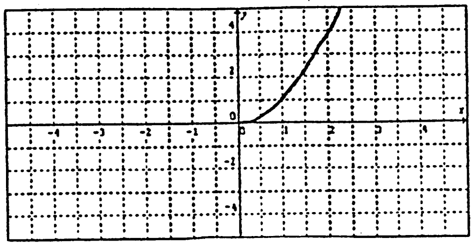
1.



2.



3.



4. a) $f(x)= 3x^2$

$f(-x)= 3(-x)^2$

$=3x^2 = f(x)$

∴ even

4. b) $f(x) = -4x^2+3x$

$f(-x) = -4(-x)^2+3(-x)$

$= -4x^2-3x \neq f(x)$

$\neq -f(x$

∴ neither

c) $f(x) = \cos x$

$f(-x) = \cos(-x)$

$= \cos(0-x)$

$= \cos 0 \cos x + \sin 0 \sin x$

$= (1)\cos x + (0)\sin x$

$= \cos x = f(x)$

∴ even

d) $f(x) = -\sin x$

$f(-x) = -\sin(-x)$

$= -\sin(0-x)$

$= -[\sin 0 \cos x - \cos 0 \sin x]$

$= -[(0)\cos x - (1)\sin x]$

$= \sin x = -f(x)$

∴ odd

e) $f(x) = |3x|$

$f(-x) = |3(-x)|$

$= |3x| = f(x)$

∴ even

f) $f(x) = 7$

$f(-x) = 7 = f(x)$

∴ even

5. a) i) $y = x^2$

replace x with $(-x)$

$y = (-x)^2$

$y = x^2$

∴ symmetric to y-axis

ii) $x = y^2$

$(-x) = y^2$

$-x = y^2$

∴ not symmetric to y-axis

iii) $x^2+y^2=1$

$(-x)^2+y^2=1$

$x^2+y^2=1$

∴ symmetric to y-axis

iv) $x^2+x^4=y$

$(-x)^2+(-x)^4=y$

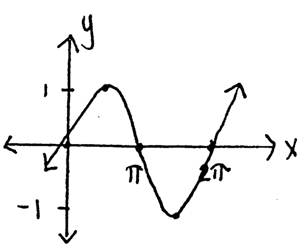
$x^2+x^4=y$

∴ symmetric to y-axis

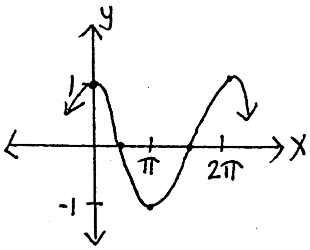
Exercise 9: Symmetry, Reflections, and Inverses (continued)

5. b) Replace (y) with (-y). If there is no change in x, then the graph is symmetric with respect to the x-axis.

6. a)



b)



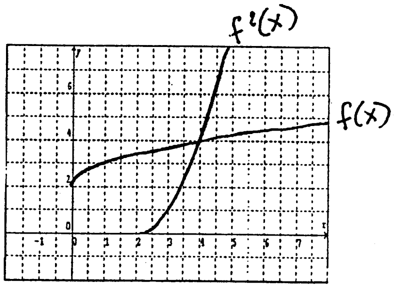
∴ not symmetric to either axis

∴ symmetric with respect to y-axis

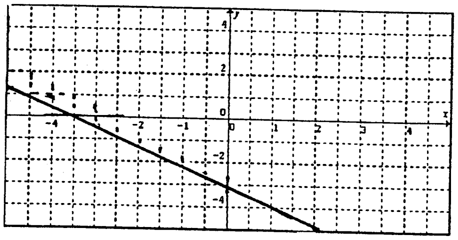
7. a) Replace y with -y:
$-y = 2x + 4$
$y = -2x - 4$

b) Replace x with -x:
$y = 2(-x) + 4$
$y = -2x + 4$

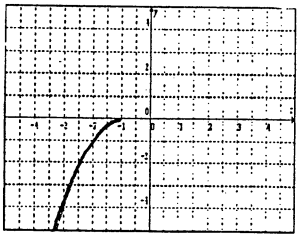
8. $f(x) = \sqrt{x} + 2$
$y = \sqrt{x} + 2$
Switch $x \& y$:
$x = \sqrt{y} + 2$
$x - 2 = \sqrt{y}$
$(x-2)^2 = y$
$\therefore f^{-1}(x) = (x-2)^2, x \geq 0$



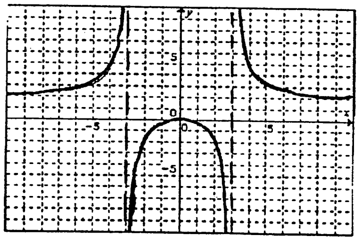
9. a)



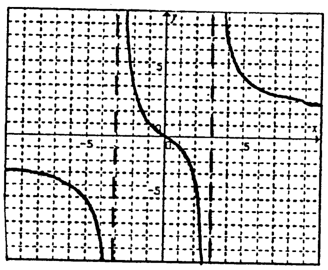
b)



10. a)



b)



Exercise 9: Symmetry, Reflections, and Inverses (continued)

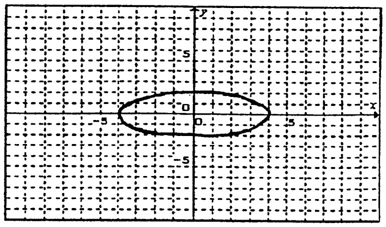
11. $\sec\theta = -2$

$\cos\theta = -\frac{1}{2}$

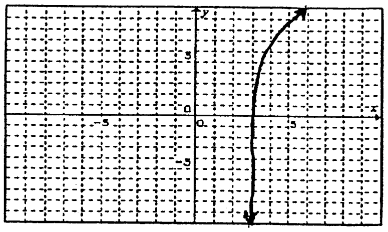
$\theta = 1.047197551$ - related angle

$\theta = 2.09 + 2k\pi,\ 4.19 + 2k\pi$, where k is an integer.

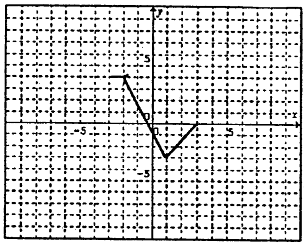
12. a)



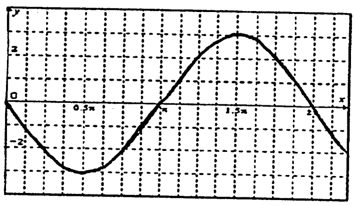
b)



c)



13.



14. $\cos^2\theta + \sin^2\theta + 3\sin\theta = 3$

$1 + 3\sin\theta = 3$

$3\sin\theta = 2$

$\sin\theta = \frac{2}{3}$

$\theta = 0.7297 + 2k\pi,\ 2.4119 + 2k\pi,$ where $k \in I$

15. $d = \dfrac{|0-2(0)+15|}{\sqrt{1^2+2^2}}$

$= \dfrac{15}{\sqrt{5}} = \dfrac{15\sqrt{5}}{5} = 3\sqrt{5}$

16. $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{\sqrt{8}}{9}\right)^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - \frac{8}{81}$

$\cos^2\theta = \frac{73}{81}$

$\cos\theta = \pm\frac{\sqrt{73}}{9}$

Since $0 < \theta < \frac{\pi}{2}$, $\cos\theta = \frac{\sqrt{73}}{9}$

$\therefore P(2\theta) = \left(\frac{65}{81}, \frac{4\sqrt{146}}{81}\right)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{\sqrt{8}}{9}\right)\left(\frac{\sqrt{73}}{9}\right) = \frac{2\sqrt{584}}{81}$

$= \frac{4\sqrt{146}}{81}$

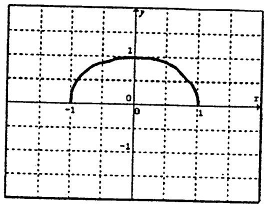
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{\sqrt{73}}{9}\right)^2 - \left(\frac{\sqrt{8}}{9}\right)^2$

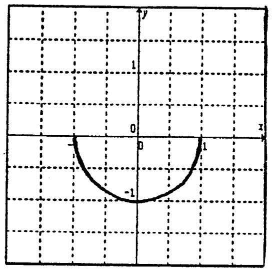
$= \frac{73-8}{81} = \frac{65}{81}$

Exercise 9: Symmetry, Reflections, and Inverses (continued)

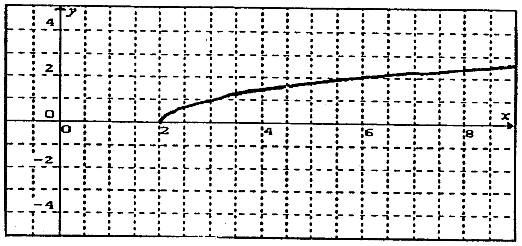
17. a)



b)



18.



This is a semi-parabola.

19. $y = -(x-1)(x-3)$ or $y = -2(x-1)(x-3)$
Any parabola of the form $y = a(x-1)(x-3)$
where a is any negative real number.

20. $y = a(x-1)(x-3)$ and passes through $(2, 16)$

$\therefore 16 = a(2-1)(2-3)$

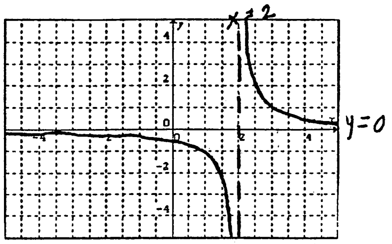
$16 = a(1)(-1)$

$-16 = a$

$\therefore y = -16(x-1)(x-3)$

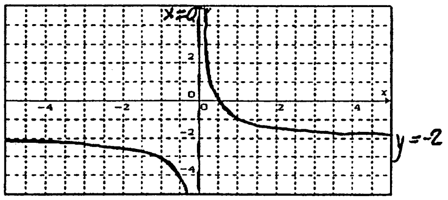
Exercise 10: Graphing Reciprocals

1.a)



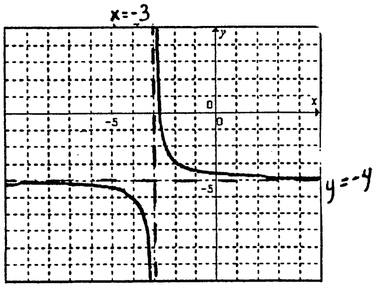
Domain: $\{x \,|\, x \neq 2\}$
Range: $\{y \,|\, y \neq 0\}$
zeroes : none

b)



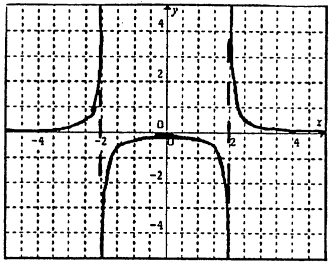
Domain : $\{x \,|\, x \neq 0\}$
Range : $\{y \,|\, y \neq -2\}$
zeroes : $1/2$

2)



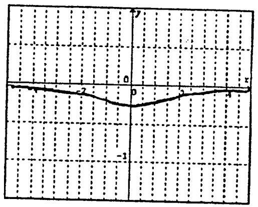
Domain : $\{x \,|\, x \neq -3\}$
Range : $\{y \,|\, y \neq -4\}$
zeroes : $-1/4$

3.



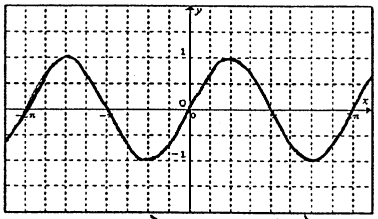
Domain : $\{x \,|\, x \neq 2 \text{ or } x \neq -2\}$
Range : $(-\infty, -1/4] \cup (0, \infty)$
Zeroes : none

4.



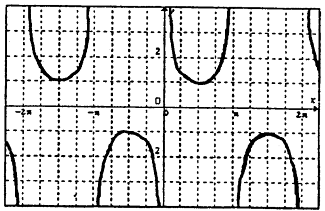
Domain : $\{x \in R\}$
Range: $\{y \,|\, -1/4 \leq y < 0\}$
zeroes : none

5. a)



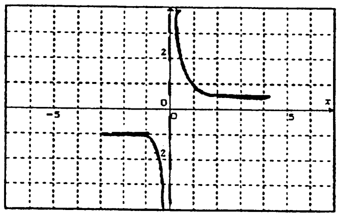
Domain : $\{x \in \text{Reals}\}$
Range: $\{y \,|\, y - 1 \leq y \leq 1, y \in \text{Reals}\}$
zeroes : $K\pi$, $K \in$ Integers

5 b)



Domain: $\{x \in \text{Reals} \,|\, x \neq n\pi, n \in \text{Int.}\}$
Range: $\{y \,|\, y \leq -1 \text{ or } y \geq 1, y \in \text{Reals}\}$
zeroes: none

6.



Exercise 10: Graphing Reciprocals (continued)

7. $\tan^2\theta + 4\sin\theta = \sec^2\theta - 2$

$\sec^2\theta - 1 + 4\sin\theta = \sec^2\theta - 2$

$4\sin\theta = -1$

$\sin\theta = -\frac{1}{4}$

θ is in III, IV

$\theta = 0.2527$ - related angle

$\theta = 3.3943 + 2k\pi,\ 6.0305 + 2k\pi,$

where $k \in I$

8. $\cos\theta > 0 \Rightarrow$ I or IV

$\tan\theta < 0 \Rightarrow$ II or IV

∴ Quadrant IV

9. $\sin^2\theta + \cos^2\theta = 1$

$(\frac{1}{2})^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - \frac{1}{4}$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

since θ is in Quad II,

$\cos\theta = -\frac{\sqrt{3}}{2}$ and ∴ $\sec\theta = -\frac{2}{\sqrt{3}}$

≈ 1.1547

10. Vertical asymptotes ⇒ the equation is of the form $x = k$

The asymptotes are at $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = -\frac{3\pi}{2}$

∴ ODD integral multiples of $\frac{\pi}{2}$

∴ "C" is answer.

11. a) $\frac{\frac{3\pi}{2}}{\pi} = \frac{x}{180}$

$x = 270°$

b) $\frac{57}{\pi} = \frac{x}{180}$

$x = 10260\pi°$

or $57 \times \frac{180}{\pi} = 3625.86°$

c) $\frac{-8.5}{\pi} = \frac{x}{180}$

$x = -1530\pi°$

or $-8.5 \times \frac{180}{\pi} = -487.01°$

d) $\frac{-22\pi}{\pi} = \frac{x}{18}$

$x = -3960°$

or $-22\pi \times \frac{180}{\pi} = -3960°$

12. $2\tan\alpha - 4\cos\beta = 4$ ①

$\tan\alpha + 2\cos\beta = -1$ ②

②×2 $2\tan\alpha + 4\cos\beta = -2$

① $2\tan\alpha - 4\cos\beta = 4$

(add) $4\tan\alpha = 2$

$\tan\alpha = \frac{1}{2}$

$\alpha = 0.4636$ or 3.605

$2\tan\alpha + 4\cos\beta = -2$

$2\tan\alpha - 4\cos\beta = 4$

(subtract) $8\cos\beta = -6$

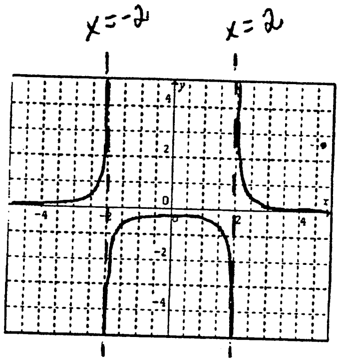
$\cos\beta = -\frac{3}{4}$

ref. angle: $\beta = 0.7227$

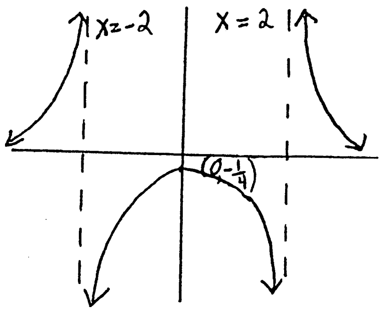
$\beta = 2.419$ or 3.864

Exercise 10: Graphing Reciprocals (continued)

13.



14. $f(x) = \frac{1}{x^2-4}$



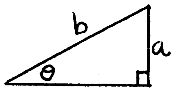
$g(x) = k \Rightarrow$ horizontal line.

a) No solution implies the line is between the x-axis and the vertex of the parabola $\therefore -\frac{1}{4} < k \le 0$

b) One solution implies the line passes through the vertex of the parabola $\therefore k = -\frac{1}{4}$

c) More than one solution occurs every where else $\therefore k < -\frac{1}{4}$ or $k > 0$

15. Solution One:



Construct a right triangle with hypotenuse "b" and one leg of length "a".

Let θ be the angle opposite a.

By Pythagorus, the other leg is $\sqrt{b^2-a^2}$ $\therefore \tan\theta = \frac{a}{\sqrt{b^2-a^2}}$

Solution Two:

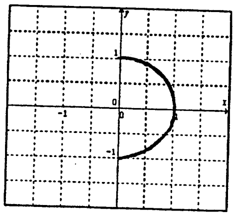
$\sin\theta = \frac{a}{b}$

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-a^2/b^2}$

$= \sqrt{\frac{b^2-a^2}{b^2}} = \frac{\sqrt{b^2-a^2}}{b}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a/b}{\sqrt{b^2-a^2}/b} = \frac{a}{\sqrt{b^2-a^2}}$

16.



17. The equation is of the form $y = a(x+1)(x-7)$. Since it passes through $(3,-6)$

$$-6 = a(3+1)(3-7)$$
$$-6 = a(4)(-4)$$
$$-6 = -16a$$
$$a = 3/8$$
$$\therefore y = 3/8(x+1)(x-7) = 3/8 x^2 - 9/4 x - 21/8$$

Exercise 10: Graphing Reciprocals (continued)

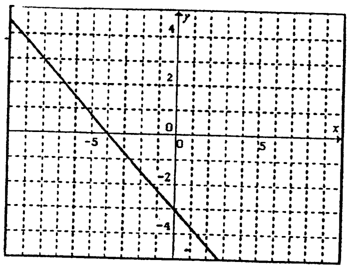
18. $\sin\theta + 2\sin\theta\cos\theta = 0$

$\sin\theta(1 + 2\cos\theta) = 0$

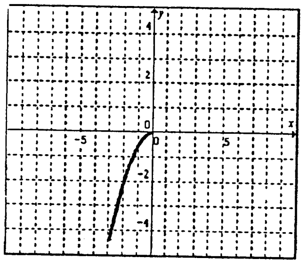
$\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\theta = 0, \pi, 2\pi$ $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

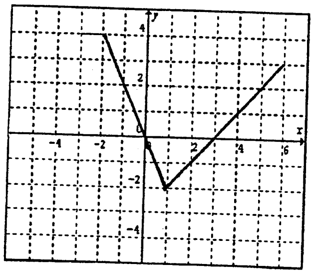
19. a)



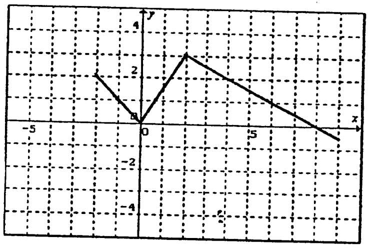
b)



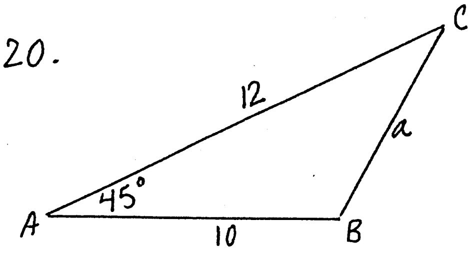
c)



d)



20.



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 12^2 + 10^2 - 2(12)(10)\cos 45$

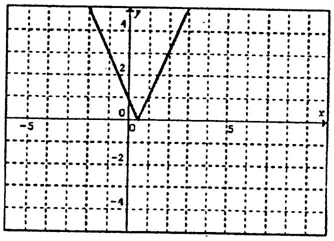
$a^2 = 244 - 240\left(\frac{\sqrt{2}}{2}\right)$

$a^2 = 244 - 120\sqrt{2}$

$a = \sqrt{244 - 120\sqrt{2}}$ or $2\sqrt{61 - 30\sqrt{2}}$

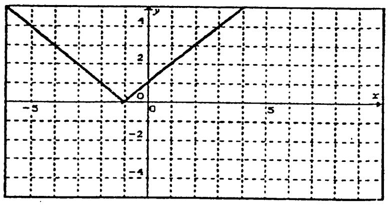
Exercise 11: Graphing Absolute Values

1.



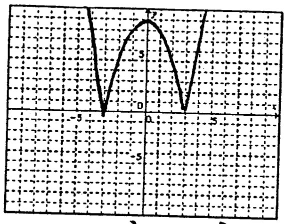
Domain : $\{x \in R\}$
Range : $\{y | y \geq 0\}$
zeroes : $1/2$

2.



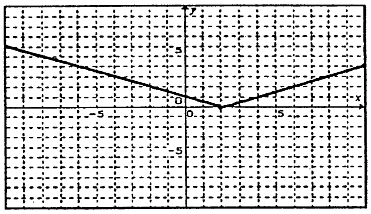
Domain : $\{x \in R\}$
Range : $\{y | y \geq 0\}$
zeroes : -1

3.



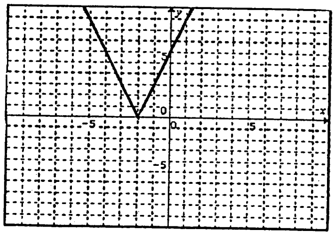
Domain : $\{x \in R\}$
Range : $\{y | y \geq 0\}$
zeroes ± 3

4.



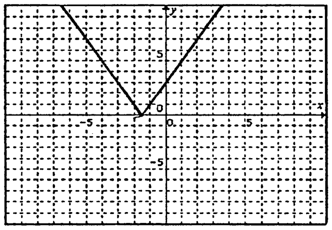
Domain : $\{x \in R\}$
Range : $\{y | y \geq 0\}$
zeroes : 2

5.



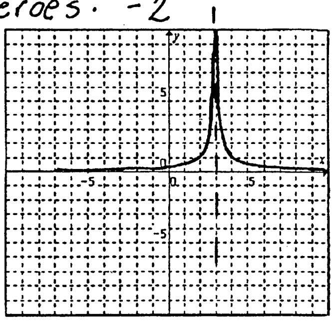
Domain : $\{x \in R\}$
Range : $\{y | y \geq 0\}$
zeroes : -2

6.



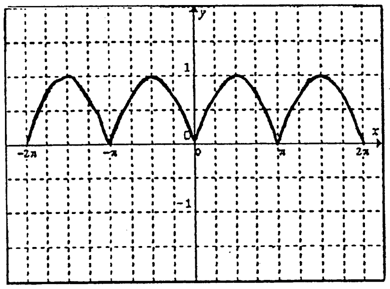
Domain : $\{x \in R\}$
Range : $\{y | y \leq 3\}$
zeroes : $\frac{-3}{2}$, $\frac{-9}{2}$

7.



Domain : $\{x | x \neq 3\}$
Range: $\{y | y > 0\}$ zeroes: none

8.

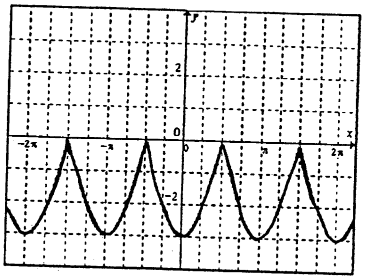


period : π

Domain: $\{x \in R\}$ Range: $\{y | 0 \leq y \leq 1\}$

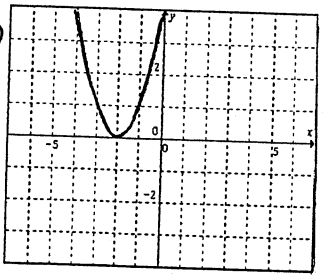
Exercise 11: Graphing Absolute Values (continued)

9.



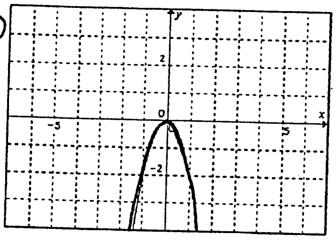
Domain : $\{x \in R\}$
Range : $\{y \mid -3 \le y \le 0\}$
Period : π

10. a)



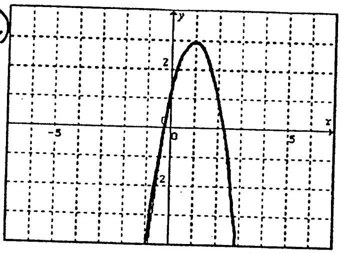
Zeroes : -2

b)



Zeroes : 0

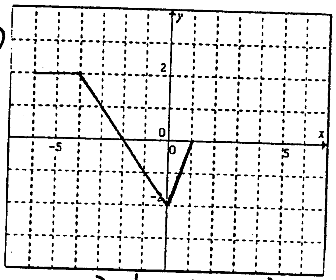
c)



Zeroes : $1 \pm \sqrt{1.5}$

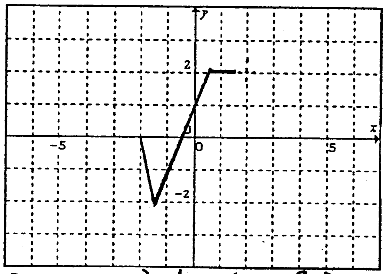
11. Range : $\{y \mid -5 \le y \le 1\}$
Period: 4π
Amplitude : 3

12. a)



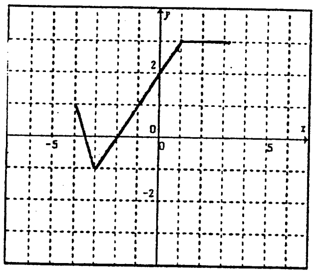
Domain : $\{x \mid -6 \le x \le 1\}$
Range : $\{y \mid -2 \le y \le 2\}$

b)

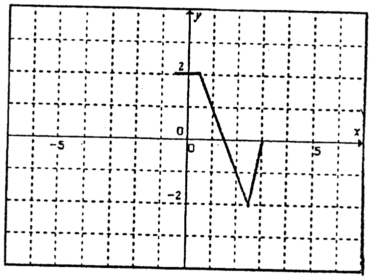


Domain : $\{x \mid -2 \le x \le \frac{3}{2}\}$
Range : $\{y \mid -2 \le y \le 2\}$

c)



d)



Domain : $\{x \mid -4 \le x \le 3\}$
Range : $\{y \mid -1 \le y \le 3\}$

Domain: $\{x \mid -\frac{1}{2} \le x \le 3\}$
Range : $\{y \mid -2 \le y \le 2\}$

Exercise 11: Graphing Absolute Values (continued)

13. $4\sin^2\theta - 8\cos\theta = -1$

$4(1-\cos^2\theta) - 8\cos\theta = -1$

$4 - 4\cos^2\theta - 8\cos\theta = -1$

$-4\cos^2\theta - 8\cos\theta = -5$

$4\cos^2\theta + 8\cos\theta - 5 = 0$

$(2\cos\theta + 5)(2\cos\theta - 1) = 0$

$2\cos\theta + 5 = 0$ or $2\cos\theta - 1 = 0$

$\cos\theta = -5/2$ $\quad$ $\cos\theta = 1/2$

no solution $\quad$ $\theta = \pi/3,\ 5\pi/3$

14. There are many graphs that could have that range.

Three examples are:

$f(x) = x^2 + 5$ [A parabola with vertex $(0, +5)$]

$f(x) = |x-3| + 5$ [An absolute value function with minimum at $(3, +5)$]

$f(x) = \sqrt{5x-3} + 5$ [A square root function that starts at $(3/5, +5)$]

15. $\angle PAB = \theta/2$, $\therefore \tan\theta/2 = PB/1+OB$

$(PB)^2 + (OB)^2 = 1$, $\therefore (PB)^2 = 1-(OB)^2$

$$\frac{PB}{1+OB} = \frac{(PB)(1-OB)}{1-(OB)^2} = \frac{PB(1-OB)}{PB^2} = \frac{1-OB}{PB} \quad \therefore \tan\frac{\theta}{2} = \frac{1-OB}{PB}$$

But $\cos\theta = OB$ and $\sin\theta = PB$

$\therefore \tan\theta/2 = \dfrac{1-\cos\theta}{\sin\theta}$ or use the $1-\cos^2\theta = \sin^2\theta$ identity.

16. $2 = a(2)^2 + c \Rightarrow 2 = 4a + c$

$-3 = a(1)^2 + c \Rightarrow -3 = a + c$

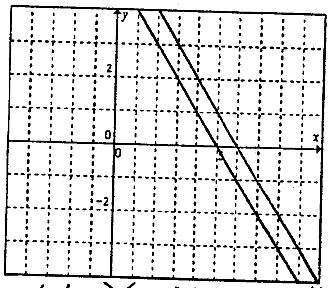
$5 = 3a$

$a = 5/3$

$-3 = 5/3 + c$

$-14/3 = c$

17.



$5 \le x + y \le 6 \Rightarrow (x,y)$ lies on the strip between the lines $x+y=5$ and $x+y=6$

Exercise 11: Graphing Absolute Values (continued)

18. $\frac{3}{x^2+x} + \frac{1}{x} = 1$

$\frac{3}{x(x+1)} + \frac{1}{x} = 1$

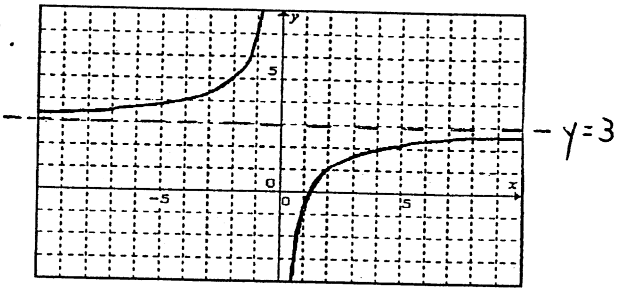
$3 + x + 1 = x(x+1)$

$4 + x = x^2 + x$

$4 = x^2$

$x = \pm 2$

19.



20. Solve: $\begin{cases} y = 2 - x \\ y = x^2 - 2x + 3 \end{cases} \Rightarrow \begin{array}{l} 2 - x = x^2 - 2x + 3 \\ 0 = x^2 - x + 1 \end{array}$

Since $b^2 - 4ac = (-1)^2 - 4(1)(1) = -3 < 0$, there are no solutions. The line does not cross the parabola.

Exercise 12: Practice with Transformations

1 a) vertical stretch by a factor of 4 b) reflect in x-axis

2. a) vertical compression by a factor of 4 b) vertical stretch, reflect in x-axis

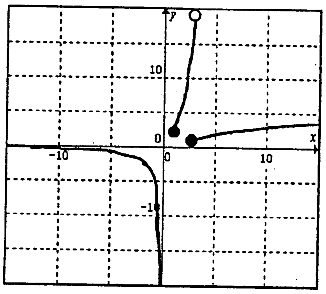
3. a) vertical compression, reflect in x-axis b) horizontal compression

4. a) vertical stretch, shift 1 unit up b) reflect in x-axis, shifts 6 units up

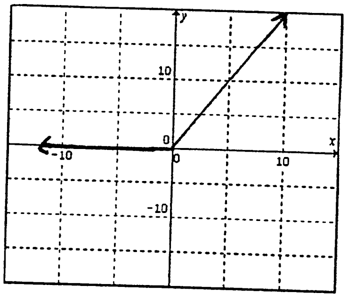
5. a) shift 1 unit left, vertical stretch b) shift 2 units right, vertical stretch

6. a) vertical compression, shift 5 units down b) vertical compression shift 4 units up.

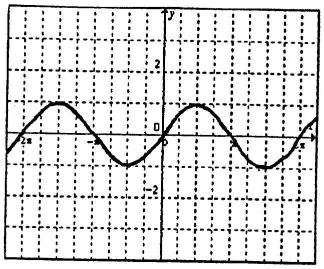
7.



8.

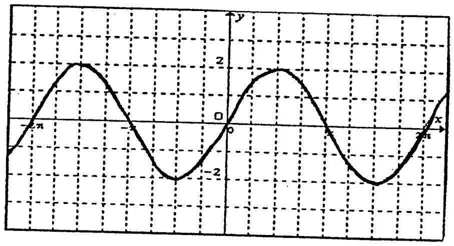


9. a)



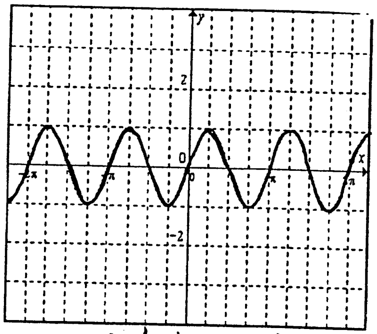
Range. $\{y \mid -1 \le y \le 1\}$ Period: 2π
Amplitude: 1

b)



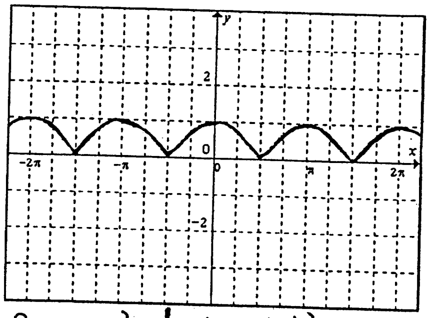
Range: $\{y \mid -2 \le y \le 2\}$ Period: 2π
Amplitude: 2

c)



Range: $\{y \mid -1 \le y \le 1\}$
Period: π
Amplitude: 1

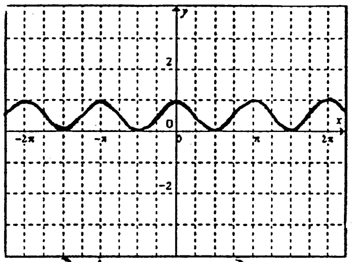
10. a)



Range $\{y \mid 0 \le y \le 1\}$
Period: π

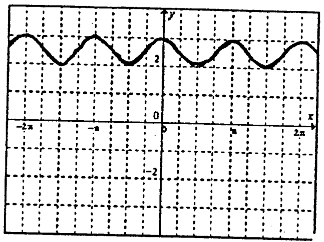
Exercise 12: Practice with Transformations (continued)

10. b)



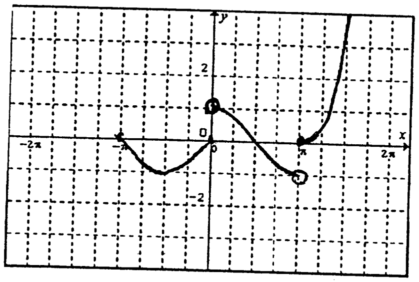
Range: $\{y \mid 0 \le y \le 1\}$
Period: π

c)



Range: $\{y \mid 2 \le y \le 3\}$
Period: π

11. $f(x) = \begin{cases} \sin x & \text{on } [-\pi, 0] \\ \cos x & \text{on } (0, \pi) \\ (x-\pi)^3 & \text{on } [\pi, \infty) \end{cases}$



12. (h, k) is the centre
radius $= h$ since circle is tangent to the y-axis.

$\therefore k = 7 - h$

$h = \dfrac{|2(h) - (7-h) - 2|}{\sqrt{2^2 + 1^2}}$

$h = \dfrac{|2h - 7 + h - 2|}{\sqrt{5}}$

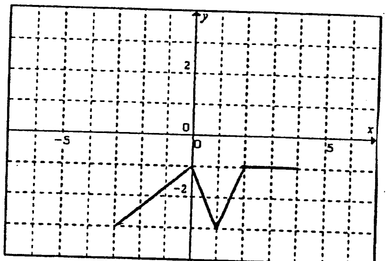
$\sqrt{5}h = 3h - 9$ (centre in 1st Quad)

$3h - \sqrt{5}h = 9$

$h = \dfrac{9}{3 - \sqrt{5}} \approx 1.72$

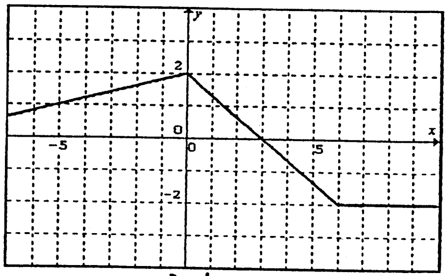
$\therefore (h, k) = (1.72, 5.28)$

13.



Domain: $\{x \mid -3 \le x \le 4\}$
Range: $\{y \mid -3 \le y \le -1\}$

14.

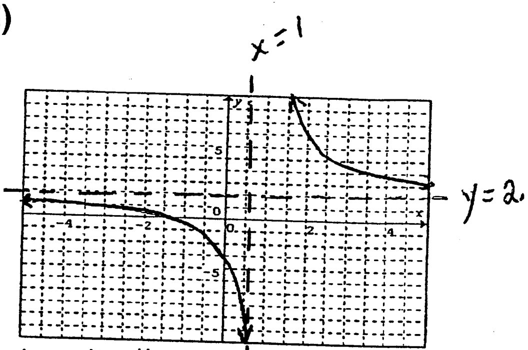


Domain: $\{x \mid -9 \le x \le 12\}$
Range: $\{y \mid -2 \le y \le 2\}$

Exercise 12: Practice with Transformations (continued)

15. $$\begin{array}{r} 2 \\ x-1\overline{)\,2x+3} \\ \underline{2x-2} \\ 5 \end{array} \qquad y = 2 + \frac{5}{x-1}$$

16.



Start with the graph of $y = 1/x$,
Multiply by 2 to stretch vertically,
shift it right 1 unit, and 2 units up.

17. Edge length of square = $P/4$

$\therefore \pi r^2 = \left(\frac{P}{4}\right)^2$

$r^2 = \frac{P^2}{16\pi}$

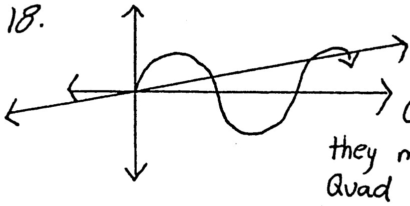
$r = \frac{P}{4\sqrt{\pi}}$

$C = 2\pi r = \frac{2\pi P}{4\sqrt{\pi}}$

$= \frac{\sqrt{\pi}\,P}{2}$

$\therefore$ (C) is the answer

18.



The graphs meet at 3 points in Quad I. By symmetry they meet at 3 points in Quad 3, and at the origin. That is a total of 7 points.
$\therefore$ (E) is the answer.

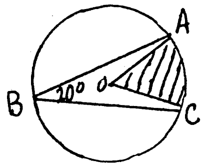
19. $\sqrt{169-144} = \sqrt[n]{125}$

$\sqrt{25} = \sqrt[n]{125}$

$5 = \sqrt[n]{125}$

$n = 3$

20.



Diameter of circle = 20 dm

Area of circle = 100π dm^2

Since $\angle B = 20°$, $\angle O = 40°$

Area of shaded region =

$100\pi \cdot \frac{40}{360} = 34.91 \text{ dm}^2$

Exercise 13: Transformations with Trig Functions

1. Amplitude : 5 $\Rightarrow y = 5\sin(\frac{1}{2}x)$
Period : 4π

2. Amplitude: 5 $\Rightarrow y = 5\cos[\frac{1}{2}(x-\pi)]$
Period : 4π
Phase Shift: π units right

3. Amplitude: 3
Period : π $\Rightarrow y = 3\sin[2(x+\pi/4)]$ or $y = -3\sin[2(x-\frac{\pi}{4})]$
Phase shift : $\pi/4$ units left

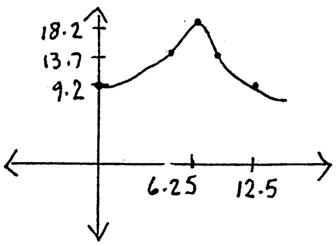
4. Amplitude: 3 $\Rightarrow y = 3\cos 2x$
Period : π

5. Amplitude : 2
Period: $\frac{4\pi}{3}$

Horizontal Translation : 2

Sine Curve : No phase shift $b = \frac{2\pi}{\frac{4\pi}{3}} = \frac{3}{2}$
$\Rightarrow y = 2\sin(3/2\, x) + 2$
Cosine Curve: Phase shift : $\pi/3$ units right
$\Rightarrow y = 2\cos[\, 3/2(x - \pi/3)] + 2$

6. a)



b) Period $= \frac{2\pi}{0.16\pi} = 12.5$ hours

c) Solve: $14.5 = -4.5\cos(0.16\pi t) + 13.7$
$0.8 = -4.5\cos(0.16\pi t)$
$\cos(0.16\pi t) = -0.177778$
$0.16\pi t = 1.7495$
$t = 3.48$ hours
Point A (see graph) is at $t = 3.48$
Point B is at $t = 6.25 + (6.25 - 3.48)$
$= 9.02$. The ship is safe for $9.02 - 3.48$
$= 5.54$ hours

7. Average: $\frac{23.6 + 4.2}{2} = 13.9°$

Period 365 days
Range : $4.2 \le y \le 23.6$
Amplitude: $13.9 - 4.2 = 9.7$
Phase shift : 26 days right

$\Rightarrow y = -9.7\cos\left(\frac{2\pi}{365}(t-26)\right) + 13.9$

Exercise 13: Transformations with Trig Functions (continued)

8. May 26 is day 146

$\therefore y = -9.7 \cos\left(\frac{2\pi}{365}(146-26)\right) + 13.9 = 18.5$

9. $21 = 13.9 - 9.7 \cos\left(\frac{2\pi}{365}(t-26)\right)$

$-0.73196 = \cos\left(\frac{2\pi}{365}(t-26)\right)$

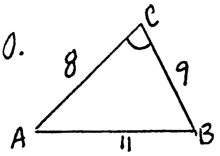
One root is: $2.39199 = \frac{2\pi}{365}(t-26)$

$\Rightarrow 139 = t - 26$

$t = 165$

The temperature reaches 21° on day 165. The maximum temperature is on day 207. The temperature will remain above 21 until day 207+(207-165) = 249. This is approximately 84 days

10.



$c^2 = a^2 + b^2 - 2ab \cos C$

$11^2 = 9^2 + 8^2 - 2(8)(9)\cos C$

$-24 = -144 \cos C$

$\cos C = 0.1666667$

$C = 80.40593177$

$c = 8.41°$

11. a) $\frac{163}{180} = \frac{x}{\pi}$

$x = 2.8449$

b) $\frac{189}{180} = \frac{x}{\pi}$

$x = 3.2987$

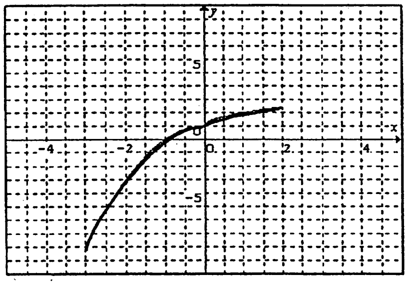
c) $\frac{216}{180} = \frac{x}{\pi}$

$x = 3.7699$

d) $\frac{352}{10} = \frac{x}{\pi}$

$x = 6.1436$

12.



13. $Mx = \frac{1}{M}\left(\frac{1}{r} + \frac{1}{P}\right)$

$M^2x = \frac{1}{r} + \frac{1}{P}$

$M^2x - \frac{1}{P} = \frac{1}{r}$

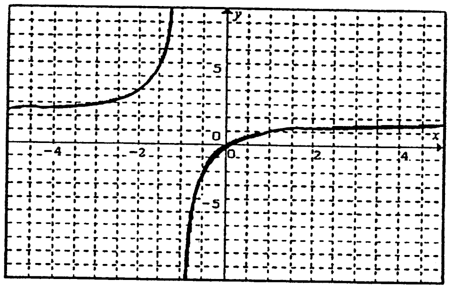
$\frac{M^2xP - 1}{P} = \frac{1}{r}$

$r = \frac{P}{m^2xP - 1}$

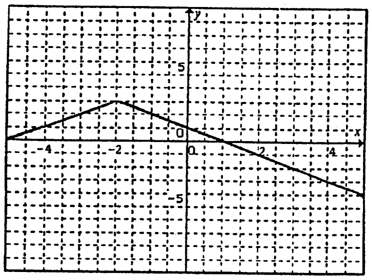
14. a) 4 b) 5 c) 3 d) 2 e) 1

Exercise 13: Transformations with Trig Functions (continued)

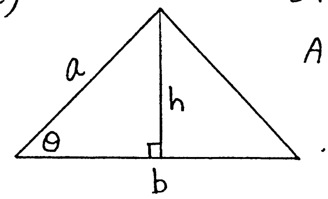
15.



16.



17. a)



$\sin\theta = \frac{h}{a} \Rightarrow h = a\sin\theta$

$A = \frac{1}{2}bh$

$\therefore A = \frac{1}{2}b(a\sin\theta)$

$= \frac{1}{2}ab\sin\theta$

b) $A = \frac{1}{2}(15)(20)\sin 45^\circ$

$= \frac{1}{2}(300)\left(\frac{\sqrt{2}}{2}\right)$

$= 75\sqrt{2}$

18. Solution 1: $y = (x^2+8x+16)-16$

$y = (x+4)^2 - 16$

vertex: $(-4, -16)$

The line $y = -16$ meets the parabola at its vertex. $\therefore K = -16$

Solution 2: $\begin{cases} y = k \\ y = x^2+8x \end{cases}$

$x^2+8x = k \Rightarrow x^2+8x-k=0$

If there is only one root,

$b^2-4ac=0 \therefore 64+4k=0, k=-16$

19. $\sqrt{x} + 2^y = 19$

$\sqrt{x} - 2^y = 3$

$2\sqrt{x} = 22$

$\sqrt{x} = 11$

$x = 121$

$11 + 2^y = 19$

$2^y = 8$

$y = 3$

20. a) Infinitely many

b) $y = x(x-2)$

$y = 5x(x-2)$

and others of the form $y = kx(x-2)$ where $k \in R$

Exercise 14: Trigonometric Identities I

1\. a) $\csc\theta = \dfrac{1}{\sin\theta}$ b) $\cos^2\theta = 1-\sin^2\theta$ c) $\cot^2\theta = \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1-\sin^2\theta}{\sin^2\theta}$

2\. a) $\sec\theta = \dfrac{1}{\cos\theta}$ b) $\sin^2\theta = 1-\cos^2\theta$ c) $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1-\cos^2\theta}{\cos^2\theta}$

3\. a) $\sin\theta\csc\theta = \sin\theta\left(\dfrac{1}{\sin\theta}\right) = 1$ b) $\sin^2\theta + \dfrac{1}{\sec^2\theta} = \sin^2\theta + \cos^2\theta = 1$

c) $1-\csc^2\theta = 1-\dfrac{1}{\sin^2\theta}$ or $\dfrac{\sin^2\theta - 1}{\sin^2\theta}$

4\. $\cos x \sec x = 1$

$L.H.S. = \cos x . \dfrac{1}{\cos x}$

$= 1$

$= R.H.S$

5\. $\csc x \sin x = 1$

$L.H.S. = \dfrac{1}{\sin x} . \sin x$

$= 1$

$= R.H.S$

6\. $\tan\theta \cot\theta = 1$

$L.H.S = \dfrac{\sin\theta}{\cos\theta} . \dfrac{\cos\theta}{\sin\theta}$

$= 1$

$= R.H.S.$

7\. $\cot\theta \sin\theta = \cos\theta$

$L.H.S. = \dfrac{\cos\theta}{\sin\theta} . \sin\theta$

$= \cos\theta$

$= R.H.S.$

8\. $\tan\theta\cos\theta = \sin\theta$

$L.H.S. = \dfrac{\sin\theta}{\cos\theta} . \cos\theta$

$= \sin\theta$

$= R.H.S$

9\. $4\sin\theta + 2\sqrt{3} = 0$

$4\sin\theta = -2\sqrt{3}$

$\sin\theta = \dfrac{-\sqrt{3}}{2}$

$\theta = \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

10\. $3\cos\theta + 2 = 0$

$3\cos\theta = -2$

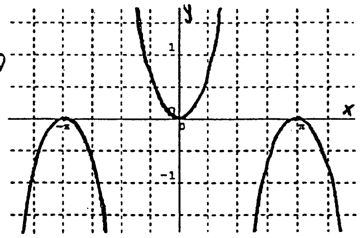
$\cos\theta = -2/3$

$\theta = 48\ 2^\circ$ - related angle

$\theta = 131.8^\circ, 228.2^\circ$

11\. a) $y = \dfrac{1}{\cos\theta} - \cos\theta$

$y = \tan\theta\sin\theta$

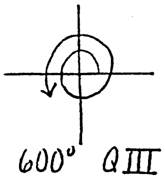


b)

$$\begin{array}{c|l} \dfrac{1}{\cos\theta} - \cos\theta & \tan\theta\sin\theta \\ \dfrac{1-\cos^2\theta}{\cos\theta} & \dfrac{\sin\theta}{\cos\theta} . \sin\theta \\ \dfrac{\sin^2\theta}{\cos\theta} & \dfrac{\sin^2\theta}{\cos\theta} \\ LHS & = RHS \end{array}$$

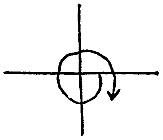
Exercise 14: Trigonometric Identities I (continued)

12. a)



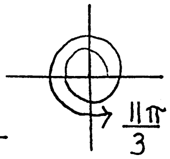
600° QIII

b)



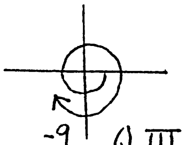
-400° QIV

c)



QIV

d)



QIII

13. $\sin^2\theta + \cos^2\theta = 1$

$(-3/4)^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - 9/16$

$\cos^2\theta = 7/16$

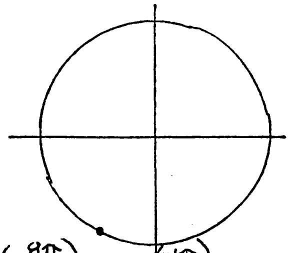
$\cos\theta = \pm\sqrt{7}/4$

$\cos\theta > 0 \Rightarrow \cos\theta = \sqrt{7}/4$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-3/4}{\sqrt{7}/4}$

$= \frac{-3}{\sqrt{7}} = -\frac{3\sqrt{7}}{7}$

14. $P\left(\frac{-8\pi}{3}\right) = \left(-\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$



$P\left(\frac{-8\pi}{3}\right) = P\left(\frac{4\pi}{3}\right)$

15. $2\cos^2\theta = \sqrt{2}\cos\theta$

$2\cos^2\theta - \sqrt{2}\cos\theta = 0$

$\cos\theta(2\cos\theta - \sqrt{2}) = 0$

$\cos\theta = 0$ or $\cos\theta = \sqrt{2}/2 = \frac{1}{\sqrt{2}}$

$\theta = \frac{3\pi}{2}$ $\quad \theta = \frac{7\pi}{4}$

16. $\cos x = -\frac{\sqrt{3}}{2}$

$x = \frac{5\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi, k \in I$

17. $\tan 2\theta = -1$

$2\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

$\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

18. $f(4) = 2$

$f(2) = 0 \therefore f(f(4)) = 0$

19. $y = \frac{3}{x+1}$

Interchange variables: $x = \frac{3}{y+1}$

$x(y+1) = 3$

$xy + x = 3$

$xy = 3 - x$

$y = \frac{3-x}{x}$

$\therefore f^{-1}(x) = \frac{3-x}{x}$

20. Solve: $\begin{cases} y = mx \\ y = x^2 + 9 \end{cases}$

$x^2 + 9 = mx$

$x^2 - mx + 9 = 0$

Since only one answer is possible,

$b^2 - 4ac = 0$

$m^2 - 36 = 0$

$m = 6$ or -6, but $M > 0 \therefore M = 6$

Exercise 15: Trigonometric Identities II

1. $(1+\sin x)(1-\sin x) = \frac{1}{\sec^2 x}$

L.H.S. $1-\sin^2 x$

$= \cos^2 x$

R.H.S $\frac{1}{\sec^2 x} = \cos^2 x$

$\therefore LHS = RHS$

2. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

Solution 1:

R.HS: $\cos^2 x + \sin^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = LHS$

Solution 2:

LHS: $1-\sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

$1 - 2\sin^2 x = RHS$

3. $\cos^2\theta = (1-\sin\theta)(1+\sin\theta)$

RHS: $1-\sin^2\theta$

$= \cos^2\theta = LHS$

4. $(1-\sec\theta)(1+\sec\theta) = -\tan^2\theta$

LHS: $1-\sec^2\theta$

$= 1-(1+\tan^2\theta)$

$= 1-1-\tan^2\theta$

$= -\tan\theta = RHS$

5. $2\sec^2 x = \frac{1}{1-\sin x} + \frac{1}{1+\sin x}$

RHS: $\frac{1+\sin x + 1 - \sin x}{1-\sin^2 x}$

$= \frac{2}{\cos^2 x} = 2\sec^2 x$

$= LHS$

6. $\sec^2 x - \csc^2 x = \tan^2 x - \cot^2 x$

LHS: $(1+\tan^2 x) - (1+\cot^2 x)$

$1+\tan^2 x - 1 - \cot^2 x$

$\tan^2 x - \cot^2 x = RHS$

7. $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

LHS: $(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$= (\cos^2 x - \sin^2 x)(1)$

$= 1-\sin^2 x - \sin^2 x$

$= 1 - 2\sin^2 x = RHS$

8. $\frac{\csc^2\theta - 1}{\csc^2\theta} = \cos^2\theta$

solution1:

LHS: $\frac{\cot^2\theta}{\csc^2\theta} = \frac{\frac{\cos^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}} = \cos^2\theta = RHS$

solution 2:

LHS: $1 - \frac{1}{\csc^2\theta} = 1 - \sin^2\theta = \cos^2\theta$

Exercise 15: Trigonometric Identities II (continued)

9. $\frac{\sec\theta - 1}{2} = 1 + \sec\theta$

$\sec\theta - 1 = 2 + 2\sec\theta$

$-3 = \sec\theta$

$\cos\theta = -1/3 \Rightarrow Q\,II$ and $Q\,III$

$\theta = 70.5$ - related angle

$\theta = 109.5^\circ,\ 250.5^\circ$

10. $3 - 6\tan\theta = \tan\theta + 6$

$-3 = 7\tan\theta$

$-3/7 = \tan\theta \Rightarrow Q\,II,\ IV$

$\theta = 0.4049$ related angle

$\theta = 2.7367,\ 5.8783$

11. $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{12}{13}\right)^2 = 1$

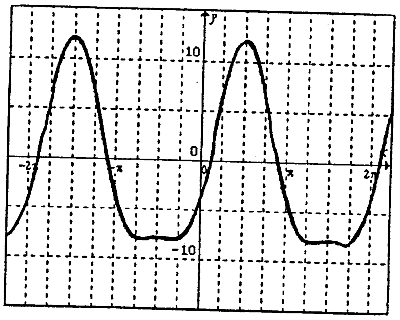
$\sin^2\theta = 1 - 144/169$

$\sin^2\theta = \frac{25}{169}$

$\sin\theta = \pm\frac{5}{13}$

$\sin\theta$ is positive since $\tan\theta > 0$ and $\cos\theta > 0$ in Quadrant I $\therefore \sin\theta = \frac{5}{13}$

12.



$\theta = 0.3398 + 2k\pi,\ 2.8018 + 2k\pi$ where k is an integer.

13. $\frac{1}{4}\sin\theta - \frac{\sqrt{3}}{8} = 0$

$2\sin\theta - \sqrt{3} = 0$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \pi/3 + 2K\pi,\ 2\pi/3 + 2K\pi$ where $K \in I$

14. 2 possible proofs:

Soln. 1. $LHS = \frac{\sec^2 t - 1}{\sec^2 t}$

$= \frac{\frac{1}{\cos^2 t} - 1}{\frac{1}{\cos^2 t}}$

$= \frac{\frac{1 - \cos^2 t}{\cancel{\cos^2 t}}}{\frac{1}{\cancel{\cos^2 t}}}$

$= \sin^2 t = RHS$

14. Solution 2. $LHS. = \frac{\sec^2 t - 1}{\sec^2 t}$

$= \frac{\tan^2 t}{\sec^2 t}$

$= \frac{\frac{\sin^2 t}{\cos^2 t}}{\frac{1}{\cos^2 t}}$

$= \sin^2 t = RHS$

Exercise 15: Trigonometric Identities II (continued)

15. a) $\frac{225}{180} = \frac{x}{\pi} \Rightarrow x = \frac{225\pi}{180}$

$x = 3.92699$

b) $\frac{216}{180} = \frac{x}{\pi} \Rightarrow x = \frac{216}{180}\pi$

$x = 3.7699$

c) $\frac{125}{180} = \frac{x}{\pi} \Rightarrow x = \frac{125}{180}\pi$

$x = 2.1817$

d) $\frac{105}{180} = \frac{x}{\pi} \Rightarrow x = \frac{105}{180}\pi$

$x = 1.8236$

16. a) $\frac{2\pi/3}{\pi} = \frac{x}{180} \Rightarrow x = \frac{2}{3}(180) = 120°$

b) $\frac{5\pi/6}{\pi} = \frac{x}{180} \Rightarrow x = \frac{5}{6}(180) = 150°$

c) $\frac{4\pi/3}{\pi} = \frac{x}{180} \Rightarrow x = \frac{4}{3}(180) = 240°$

d) $\frac{3\pi/4}{\pi} = \frac{x}{180} \Rightarrow x = \frac{3}{4}(180) = 135°$

17. $f(f(x)) = 2(2x+3)+3$

$= 4x+6+3$

$= 4x+9$

$\therefore y = 4x+9$

y-intercept : 9

18. $a < 0$, so the parabola opens down

$b^2-4ac > 0$, so the parabola crosses the x-axis in two places. $\therefore$ C

19. Let $f(x) = x^3-2x^2+3x-6$

$f(2) = 2^3-2(2)^2+3(2)-6$

$=0$

$\therefore x=2$ is a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

$\therefore f(x) = (x-2)(x^2+3)$

20. $\frac{x+3}{x-2} = 5$

$5x-10 = x+3$

$4x = 13$

$x = 13/4$

$\therefore f(13/4) = 5$

so $f^{-1}(5) = 13/4$

Exercise 16: Sum and Difference Identities I

1. $\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\pi/6 \, \sin\pi/4$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

2. $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \cos\pi/3\,\cos\pi/4 + \sin\pi/3\,\sin\pi/4$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4},$$

3. $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \begin{matrix}\div \cos\alpha\cos\beta \\ \div \cos\alpha\cos\beta\end{matrix}$

$$\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

4. a) $\sin\left(\frac{7\pi}{12}\right) = \sin(\pi/3+\pi/4) = \sin\pi/3\cos\pi/4+\cos\pi/3\sin\pi/4$

$$= (\sqrt{3}/2)(\sqrt{2}/2)+\left(\tfrac{1}{2}\right)(\sqrt{2}/2) = \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

b) $\cos(7\pi/12) = \cos(\pi/3+\pi/4) = \cos\pi/3\cos\pi/4 - \sin\pi/3\sin\pi/4$

$$= (1/2)(\sqrt{2}/2)-(\sqrt{3}/2)(\sqrt{3}/2) = \frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

c) $\tan(7\pi/12) = \dfrac{\sin(7\pi/12)}{\cos(7\pi/12)} = \dfrac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}} = \dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}\cdot\dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}} = \dfrac{2+2\sqrt{12}+6}{-4}$

$$= \frac{8+4\sqrt{3}}{-4}$$

$$= -2-\sqrt{3}$$

5. a) Since $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

and $\cot(\alpha+\beta) = \dfrac{1}{\tan(\alpha+\beta)}$

$\therefore \cot(\alpha+\beta) = \dfrac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

* Can also use the tangent sum identity.

Exercise 16: Sum and Difference Identities I (continued)

5. b) $\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$= \dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta \div \cos\alpha\cos\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta \div \cos\alpha\cos\beta}$

$= \dfrac{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}}{1 + \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

6. $\tan {}^{7\pi}/_{12} = \tan({}^{\pi}/_{3} + {}^{\pi}/_{4}) = \dfrac{\tan{}^{\pi}/_{3} + \tan{}^{\pi}/_{4}}{1-\tan{}^{\pi}/_{3}\tan{}^{\pi}/_{4}} = \dfrac{\sqrt{3}+1}{1-\sqrt{3}(1)} = \dfrac{\sqrt{3}+1}{1-\sqrt{3}} \cdot \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \dfrac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

7. $\cos 105^\circ = \cos(45^\circ+60^\circ) = \cos 45^\circ \cdot \cos 60^\circ - \sin 45^\circ \sin 60^\circ$

$= ({}^{\sqrt{2}}/_{2})({}^{1}/_{2}) - ({}^{\sqrt{2}}/_{2})({}^{\sqrt{3}}/_{2}) = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

8. α in Quad II $\quad$ β in Quad IV

$\sin\alpha = {}^{7}/_{25}$ $\quad$ $\sin\beta = {}^{-40}/_{41}$

$\cos\alpha = {}^{-24}/_{25}$ $\quad$ $\cos\beta = {}^{9}/_{41}$

a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$({}^{7}/_{25})({}^{9}/_{41}) + ({}^{-24}/_{25})({}^{-40}/_{41}) = \dfrac{63}{1025} + \dfrac{960}{1025} = \dfrac{1023}{1025}$

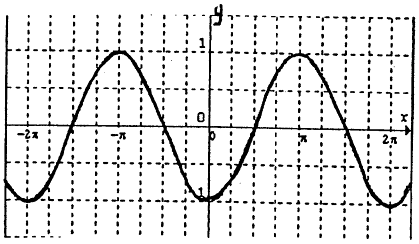
b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$({}^{-24}/_{25})({}^{9}/_{41}) - ({}^{7}/_{25})({}^{-40}/_{41}) = \dfrac{-216}{1025} + \dfrac{280}{1025} = \dfrac{64}{1025}$

c) $\sec(\alpha+\beta) = \dfrac{1}{\cos(\alpha+\beta)} = \dfrac{1025}{64}$

d) since $\sin(\alpha+\beta) > 0$ and $\cos(\alpha+\beta) > 0 \Rightarrow P(\alpha+\beta)$ is in Quad I.

9. a)



b) $LHS = \sin(t + \frac{3\pi}{2})$

$= \sin t\cos\frac{3\pi}{2} + \cos t\sin\frac{3\pi}{2}$

$= \sin t(0) + \cos t(-1)$

$= -\cos t = RHS$

Exercise 16: Sum and Difference Identities I (continued)

10. $\dfrac{\sin^2 18^\circ + \cos^2 18^\circ}{1-\cos^2 210^\circ}$

$= \dfrac{1}{\sin^2 210^\circ}$

$= \dfrac{1}{(\frac{1}{2})^2}$

$= \dfrac{1}{\frac{1}{4}} = 4$

11. $1+\tan^2\theta = \sec^2\theta$

$1+\left(\frac{6}{7}\right)^2 = \sec^2\theta$

$1+\frac{39}{49} = \sec^2\theta$

$\frac{85}{49} = \sec^2\theta$

$\pm\frac{\sqrt{85}}{7} = \sec\theta$

since θ is in QIII,

$\sec\theta = -\frac{\sqrt{85}}{7}$

12. $4\cos^2\theta + \cos\theta - 3 = 0$

$(4\cos\theta - 3)(\cos\theta + 1) = 0$

$\cos\theta = 3/4$ or $\cos\theta = -1$

$\theta = 0.7227, 5.5605$ or $\theta = 3.1416$

$\theta = 0.7227, 3.1416, 5.5605$

13. $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$

R.H.S. $\left(\dfrac{1-\cos x}{\sin x}\right)\left(\dfrac{1+\cos x}{1+\cos x}\right)$

$= \dfrac{1-\cos^2 x}{\sin x(1+\cos x)}$

$= \dfrac{\sin^2 x}{\sin x(1+\cos x)}$

$= \dfrac{\sin x}{1+\cos x} = LHS$

14. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x} = 1$

LHS: $\dfrac{\sin x}{\frac{1}{\sin x}} + \dfrac{\cos x}{1/\cos x}$

$= \sin^2 x + \cos^2 x$

$= 1 = RHS$

15. $\dfrac{1}{\csc x - \sin x} = \tan x \sec x$

LHS: $\dfrac{1}{\frac{1}{\sin x} - \sin x}$

$= \dfrac{1}{\frac{1-\sin^2 x}{\sin x}}$

$= \dfrac{\sin x}{\cos^2 x}$

$= \dfrac{\sin x}{\cos x}\left(\dfrac{1}{\cos x}\right)$

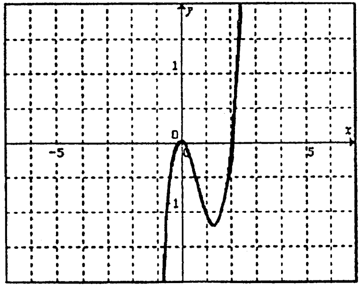
$= \tan x \sec x = RHS$

16. $P(\pi/3) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$d = \dfrac{\left|2(\frac{1}{2}) - 3(\frac{\sqrt{3}}{2}) + 7\right|}{\sqrt{2^2+-3^2}} = \dfrac{\left|1-\frac{3\sqrt{3}}{2}+7\right|}{\sqrt{4+9}}$

$= \dfrac{\frac{16-3\sqrt{3}}{2}}{\sqrt{13}} = \dfrac{16-3\sqrt{3}}{2\sqrt{13}} \approx 1.498$

17. Zeroes : 0, 2



Exercise 16: Sum and Difference Identities I (continued)

18. $\frac{x+1}{x-1} + \frac{x}{x+1} = -1$

$(x+1)(x+1) + x(x-1) = -1(x-1)(x+1)$

$x^2 + 2x + 1 + x^2 - x = -x^2 + 1$

$2x^2 + x + 1 = -x^2 + 1$

$3x^2 + x = 0$

$x(3x+1) = 0$

$x = 0$ or $x = -1/3$

19. a) A horizontal line at $y = k$ will intersect the graph in 4 places.

$\therefore 0 < k < 1$

b) A horizontal line at $y = k$ will intersect the graph in 2 places.

$\therefore$ either $k < 0$ or $k = 1$

20. a) $AB = \sqrt{6^2 + 8^2} = 10$

b) $\overset{\frown}{BC}$ is equal to the circumference of the base of the cone.

$\therefore \overset{\frown}{BC} = 2\pi(6) = 12\pi$

c) Fig. 2 is part of a circle with circumference 20π

$\therefore$ Reflex $\angle CAB = \frac{12\pi}{20\pi} \times 360 = 216°$

$\therefore \angle CAB = 144°$

d) The area of Fig. 2 is the same as the surface area of the cone.

$\therefore$ Area $= \frac{216}{360} \times \pi (10)^2 = 60\pi$

Exercise 17: Sum and Difference Identities II

1. $\cos(\pi/3+\theta) = \cos\pi/3\cos\theta - \sin\pi/3\sin\theta$

$= 1/2\cos\theta - \sqrt{3}/2\sin\theta$

$= \dfrac{\cos\theta - \sqrt{3}\sin\theta}{2}$

2. $\tan(\theta-\pi/6) = \dfrac{\tan\theta - \tan\pi/6}{1+\tan\theta\tan\pi/6}$

$= \dfrac{\tan\theta - 1/\sqrt{3}}{1+\tan\theta(1/\sqrt{3})} = \dfrac{\sqrt{3}\tan\theta - 1}{\sqrt{3}+\tan\theta}$

3. $\sec(\pi/4+\theta) = \dfrac{1}{\cos(\pi/4+\theta)}$

$= \dfrac{1}{\cos\pi/4\cos\theta - \sin\pi/4\sin\theta}$

$= \dfrac{1}{(1/\sqrt{2})\cos\theta - 1/\sqrt{2}\sin\theta}$

$= \dfrac{\sqrt{2}}{\cos\theta - \sin\theta}$

4. a) $\sin(t-\pi/2) = -\cos t$

LHS: $= \sin t\cos\pi/2 - \cos t\sin(\pi/2)$

$= \sin t(0) - \cos t(1)$

$= -\cos t = RHS$

b) $\cos(t+3\pi/2) = \sin t$

LHS: $\cos t\cos 3\pi/2 - \sin t\sin 3\pi/2$

$= \cos t(0) - \sin t(-1)$

$= \sin t$

5. a) $= \sin\left(\dfrac{5\pi}{16} - \dfrac{\pi}{16}\right) = \sin\left(\dfrac{4\pi}{16}\right) = \sin\pi/4$

$= \dfrac{\sqrt{2}}{2}$

b) $= \cos(33^\circ + 27^\circ) = \cos 60^\circ = \dfrac{1}{2}$

6. a) $\sin\alpha = -\dfrac{4}{5} \Rightarrow \cos\alpha = \dfrac{-3}{5}$

$\cos\beta = \dfrac{-5}{13} \Rightarrow \sin\beta = \dfrac{-12}{13}$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \left(\dfrac{-4}{5}\right)\left(\dfrac{-5}{13}\right) - \left(\dfrac{-3}{5}\right)\left(\dfrac{-12}{13}\right)$

$= \dfrac{20}{65} - \dfrac{36}{65} = -\dfrac{16}{65}$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \left(\dfrac{-3}{5}\right)\left(\dfrac{-5}{13}\right) + \left(\dfrac{-4}{5}\right)\left(\dfrac{-12}{13}\right)$

$= \dfrac{15}{65} + \dfrac{48}{65} = \dfrac{63}{65}$

Since $\cos(\alpha-\beta > 0$ and $\sin(\alpha-\beta) < 0 \Rightarrow$ the point $(63/65, -16/65)$ is in Quadrant IV

7. $\sin(\theta-\pi/6) + \cos(\theta-\pi/3) = \sqrt{3}\sin\theta$

LHS: $= \sin\theta\cos\pi/6 - \cos\theta\sin\pi/6 + \cos\theta\cos\pi/3 + \sin\theta\sin\pi/3$

$= \sin\theta(\sqrt{3}/2) - \cos\theta(1/2) + \cos\theta(1/2) + \sin\theta(\sqrt{3}/2)$

$= \sqrt{3}\sin\theta = RHS$

Exercise 17: Sum and Difference Identities II (continued)

8. $\tan(\theta+\pi/4)-\tan(\theta-3\pi/4)=1$

LHS: $=\frac{\tan\theta+\tan\pi/4}{1-\tan\theta\tan\pi/4}-\left(\frac{\tan\theta-\tan3\pi/4}{1+\tan\theta\tan3\pi/4}\right)$

$=\frac{\tan\theta+1}{1-\tan\theta}-\left(\frac{\tan\theta-(-1)}{1+\tan\theta(-1)}\right)=\frac{\tan\theta+1}{1-\tan\theta}-\frac{\tan\theta+1}{1-\tan\theta}$

$=\frac{0}{1-\tan\theta}=0\neq RHS$

$\therefore$ original statement is false

9. $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

LHS: $\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$=2\sin\alpha\cos\beta=RHS$

10. $\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y$

LHS: $(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)$

$=\cos^2x\cos^2y-\sin^2x\sin^2y$

$=\cos^2x(1-\sin^2y)-(1-\cos^2x)\sin^2y$

$=\cos^2x-\cos^2x\sin^2y-\sin^2y+\cos^2x\sin^2y$

$=\cos^2x-\sin^2y=RHS$

11. $\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

LHS: $\cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}$

$=\frac{\frac{1}{\tan\alpha+\tan\beta}}{1-\tan\alpha\tan\beta}=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

$=\frac{1-\left(\frac{1}{\cot\alpha}\frac{1}{\cot\beta}\right)}{\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}}=\frac{\frac{\cot\alpha\cot\beta-1}{\cot\alpha\cot\beta}}{\frac{\cot\beta+\cot\alpha}{\cot\alpha\cot\beta}}$

$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=RHS$

12. $\cos\alpha=\frac{5}{13}\Rightarrow\tan\alpha=\frac{-12}{5}$

$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$=\frac{-12/5-3/4}{1+(-12/5)(3/4)}=\frac{\frac{-48-15}{20}}{1-36/20}$

$=\frac{\frac{-63}{20}}{\frac{-16}{20}}=\frac{-63}{-16}=\frac{63}{16}$

Exercise 17: Sum and Difference Identities II (continued)

13. Possible Solutions - More do exist.

a) $\sin t = \cos(t - \pi/2)$

or $\cos(t + 3\pi/2)$

b) $\sin t = \sin(t + 2\pi)$

or $\sin(t - 4\pi)$

(any given multiple of π will work)

c) $RHS = \cos(t - \pi/2)$

$= \cos t \cos \pi/2 + \sin t \sin \pi/2$

$= \cos t(0) + \sin(t)(1)$

$= \sin t = LHS$

14. $2\sin^2\theta - 5\sin\theta - 3 = 0$

$(2\sin\theta + 1)(\sin\theta - 3) = 0$

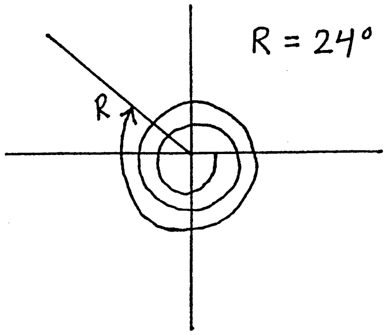
$2\sin\theta + 1 = 0$ or $\sin\theta - 3 = 0$

$\sin\theta = -1/2$ no solution

$\theta_1 = 0.52$ - related angle

$\theta = 3.66 + 2k\pi, 5.76 + 2k\pi \; k \in I$

15.



16. $3\cos^2\theta = \cos\theta$

$3\cos^2\theta - \cos\theta = 0$

$\cos\theta(3\cos\theta - 1) = 0$

$\cos\theta = 0$ or $3\cos\theta - 1 = 0$

$\cos\theta = 1/3$

$\theta = 1.57$ or 4.71

$\theta = 1.23, \; 5.05$

$\therefore \; \theta = 1.23, 1.57, 4.71, 5.05$

17. $\sin^3 x - \cos^3 x = (\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)$

18. $\tan^3 x + \cot^3 x = (\tan x + \cot x)(\tan^2 x - \tan x \cot x + \cot^2 x)$

$= (\tan x + \cot x)(\tan^2 x - 1 + \cot^2 x)$

19. $\sec^2 x(1 - \sin^2 x) = 1$

LHS: $\sec^2 x \cos^2 x = \dfrac{1}{\cos^2 x} \cdot \cos^2 x = 1 = RHS$

Exercise 18: Double Angle Identities

1. a) $\sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$
$= 2\sin\theta\cos\theta$

b) $\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$
$= \cos^2\theta - \sin^2\theta$

2. a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= \cos^2\theta - (1-\cos^2\theta)$
$= \cos^2\theta - 1 + \cos^2\theta$
$= 2\cos^2\theta - 1$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= (1-\sin^2\theta) - \sin^2\theta$
$= 1 - \sin^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

3. $\tan 2\theta = \dfrac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta}$
$= \dfrac{2\tan\theta}{1-\tan^2\theta}$

4. $\tan 120° = \tan(2(60°))$
$= \dfrac{2\tan 60°}{1-\tan^2 60°} = \dfrac{2(\sqrt{3})}{1-(\sqrt{3})^2} = \dfrac{2\sqrt{3}}{-2} = -\sqrt{3}$

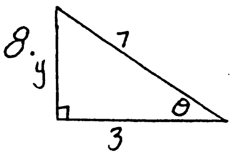
5. $\cos \pi/5 = \cos[2(\pi/10)] = 2\cos^2(\pi/10) - 1 = 2(0.95)^2 - 1 = 0.805 = 0.81$

6. LHS: $\csc 2x - \cot 2x$
$= \dfrac{1}{\sin 2x} - \dfrac{\cos 2x}{\sin 2x}$
$= \dfrac{1-\cos 2x}{\sin 2x}$
$= \dfrac{1-(\cos^2 x - \sin^2 x)}{2\cos x \sin x}$
$= \dfrac{1-\cos^2 x + \sin^2 x}{2\cos x \sin x}$
$= \dfrac{\sin^2 x + \sin^2 x}{2\cos x \sin x}$
$= \dfrac{2\sin^2 x}{2\cos x \sin x}$
$= \dfrac{\sin x}{\cos x} = \tan x = RHS$

7. a) $\sin 2\theta = 2\sin\theta\cos\theta$
$= 2(4/5)(3/5)$
$= \dfrac{24}{25}$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= \left(\dfrac{3}{5}\right)^2 - \left(\dfrac{4}{5}\right)^2$
$= \dfrac{9}{25} - \dfrac{16}{25} = -\dfrac{7}{25}$

8.



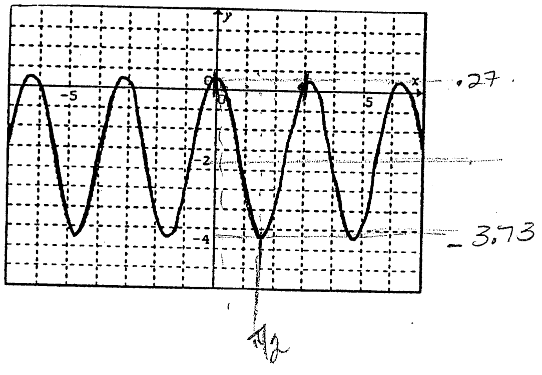
$3^2 + y^2 = 7^2$
$y^2 = 49 - 9$
$y^2 = 40$
$y = \sqrt{40}$
$y = 2\sqrt{10}$

$\sin\theta = \dfrac{2\sqrt{10}}{7}$
$\cos\theta = \dfrac{3}{7}$
$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2\left(\dfrac{2\sqrt{10}}{7}\right)\left(\dfrac{3}{7}\right)$
$= \dfrac{12\sqrt{10}}{49}$

Exercise 18: Double Angle Identities (continued)

9. Method #1
"Intersection"
$y = \cos 2\theta$, $y = \sqrt{3}/2$

Method #2
"Finding zeroes"
$y = 2\cos 2\theta - \sqrt{3}$



$\theta = 0.262,\ 2.880,\ 3.403,\ 6.021$

10. LHS: $\dfrac{(\csc^2 x - \cot^2 x)(\csc^2 x + \cot^2 x) + \cot^2 x}{\csc^2 x + \cot^2 x}$

$= \csc^2 x - \cot^2 x + \cot^2 x$

$= \csc^2 x = RHS$

11. $\cos(-\theta) = \cos(0-\theta)$

$= \cos 0 \cos\theta + \sin 0 \sin\theta$

$= (1)\cos\theta + 0(\sin\theta)$

$= \cos\theta$

$\therefore \cos\theta$ is an even function.

12. $\sin(-\theta) = \sin(0-\theta)$

$= \sin 0 \cos\theta - \cos 0 \sin\theta$

$= (0)\cos\theta - (1)\sin\theta$

$= -\sin\theta$

$\therefore \sin\theta$ is an even function.

13. $\sin^2\theta + \cos^2\theta = 1$

$(-2/3)^2 + \cos^2\theta = 1$

$4/9 + \cos^2\theta = 1$

$\cos^2\theta = 5/9$

$\cos\theta = \pm\sqrt{5}/3$

Since θ is in Quadrant IV

$\cos\theta = \sqrt{5}/3$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{5}/3}{-2/3} = \dfrac{\sqrt{5}}{2}$

14. $\dfrac{\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3}}{\sin 2\pi/3 \cdot \cos 2\pi/3}$

$= \dfrac{\sqrt{3}/2 + (-1/2)}{(\sqrt{3}/2)(-1/2)}$

$= \dfrac{\frac{\sqrt{3}-1}{2}}{\frac{-\sqrt{3}}{4}}$

$= \dfrac{2\sqrt{3}-2}{-\sqrt{3}}$

$= \dfrac{2\sqrt{3}-6}{3}$

15. LHS $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = \dfrac{(\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x)}{\sin x + \cos x}$

$= \sin^2 x + \cos^2 x - \sin x \cos x$

$= 1 - \sin x \cos x = RHS$

Exercise 18: Double Angle Identities (continued)

16. $\tan\theta = \sqrt{3}$

$\theta = \pi/3 + 2K\pi,\ 4\pi/3 + 2K\pi$

where K is an integer <u>or</u>

$\pi/3 + K\pi$, where K is an integer

17. $\csc^2\theta \sin^2\theta + 3\sin\theta + 3 = 5$

$1 + 3\sin\theta = 2$

$3\sin\theta = 1$

$\sin\theta = 1/3$

$\theta = 0.3398 + 2K\pi,\ 2.8018 + 2K\pi$

where K is an integer

18. a) $\sin(\pi/2 - x) = \sin\pi/2 \cos x - \cos\pi/2 \sin x$

$= (1)\cos x - (0)\sin x$

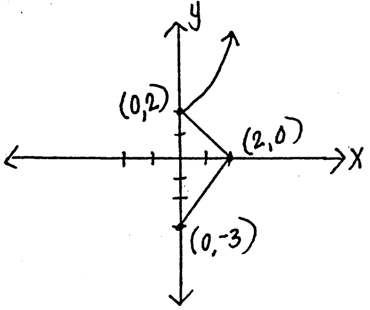
$= \cos x$

b) $\cos(\pi/2 - x) = \cos\pi/2 \cos x + \sin\pi/2 \sin x$

$= (0)\cos x + (1)\sin x$

$= \sin x$

19. a)



b) since $y = f^{-1}(x)$ is not a function, $y = f(x)$ is not one-to-one.

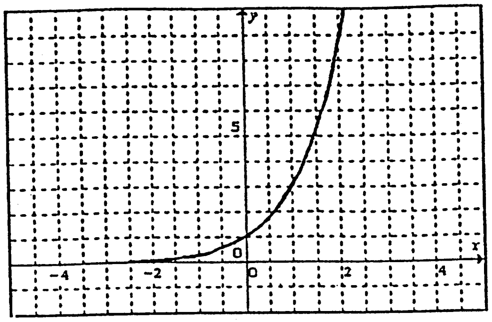
20. a) quadratic function

b) (4, 2)

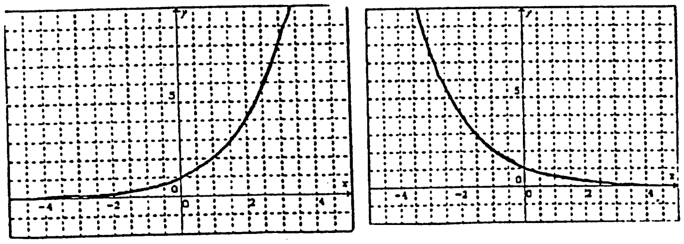
c) four units to the right and two units up.

Exercise 19: Exponential Functions

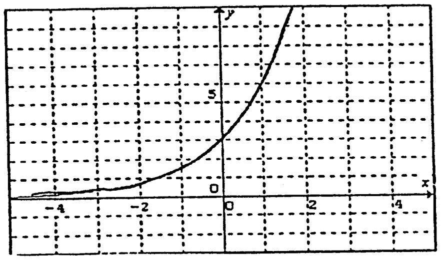
1.a)



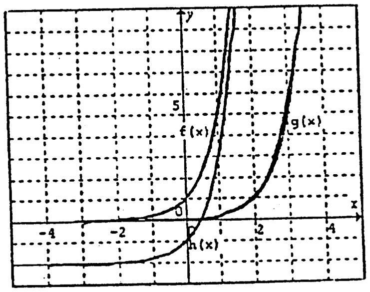
2.a)



3.



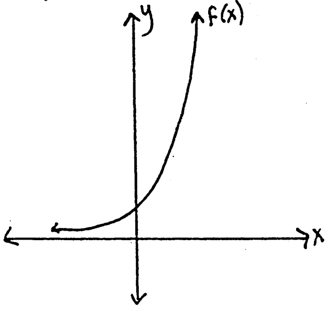
4.



5. a) Since $2^1=2$ and $2^2=4$

$\Rightarrow 2^{1.3}\approx 2.5$

b) Since $2^5=32$, $2^{5.1}\approx 34.3$



6. a) Since $3^1=3$ and $3^2=9$,

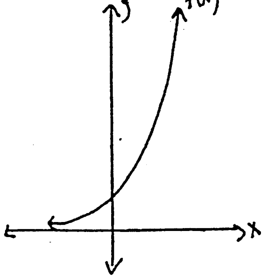
if $3^x=6 \Rightarrow 1<x<2$

$x\approx 1.6$

b) Since $3^2=9$ and $3^3=27$,

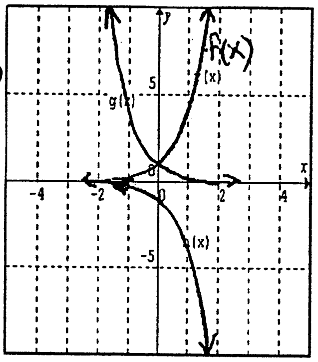
if $3^x=10 \Rightarrow 2<x<3$

$\therefore x\approx 2.1$



Exercise 19: Exponential Functions (continued)

7. a)

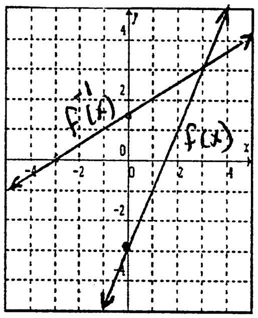


b) $f(x) = 4^x$: Increasing function
y-intercept is 1 Range: $\{y|y>0\}$

$g(x) = 4^{-x}$: Decreasing Function
y-intercept is 1 Range: $\{y|y>0\}$

$h(x) = -4^x$: Decreasing function
y-intercept is −1 Range: $\{y|y<0\}$

8. a)



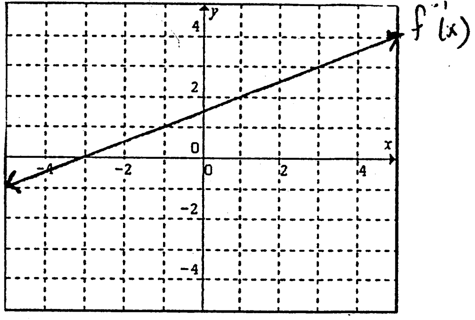
b) $\overset{f(x)}{y} = 2x - 3$

Inverse: $x = 2y - 3$

$x + 3 = 2y$

$y = \frac{x}{2} + \frac{3}{2}$

$\therefore f^{-1}(x) = \frac{x+3}{2}$



9. L.H.S: $\frac{1 - \cos^2\theta}{1 + \tan^2\theta}$

$= \frac{\sin^2\theta}{\sec^2\theta}$

$= \sin^2\theta \cos^2\theta = RHS$

10. $2\tan^2\theta + \sec\theta = 1$

$2(\sec^2\theta - 1) + \sec\theta - 1 = 0$

$2\sec^2\theta - 2 + \sec\theta - 1 = 0$

$2\sec^2\theta + \sec\theta - 3 = 0$

$(2\sec\theta + 3)(\sec\theta - 1) = 0$

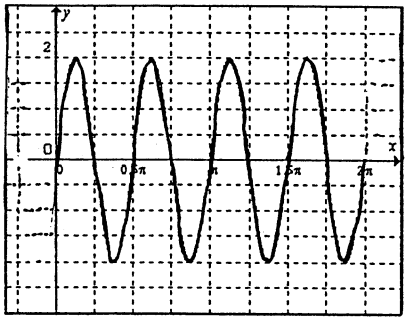
$\sec\theta = -\frac{3}{2}$ or $\sec\theta = 1$

$\cos\theta = -\frac{2}{3}$ or $\cos\theta = 1$

$\theta = 0.84107$ - related angle

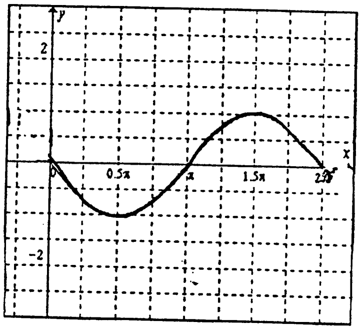
$\theta = 2.3005,\ 3.9827,\ 0,\ 2\pi$

11.



Amplitude: 2
Period: $\pi/2$

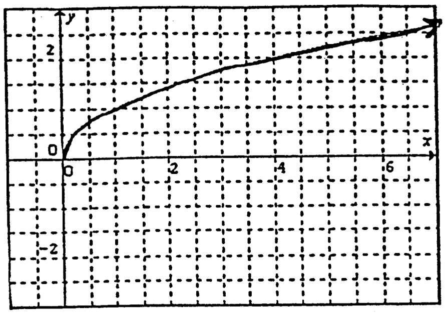
12.



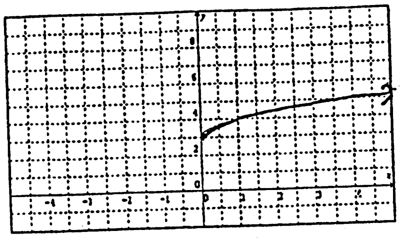
Amplitude: 1
Period: 2π
Phase shift: $\pi/2$ right

Exercise 19: Exponential Functions (continued)

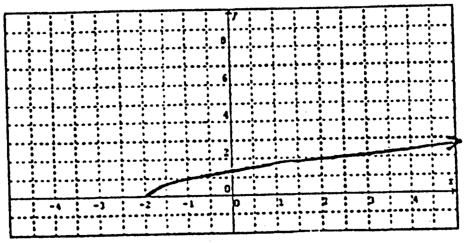
13.



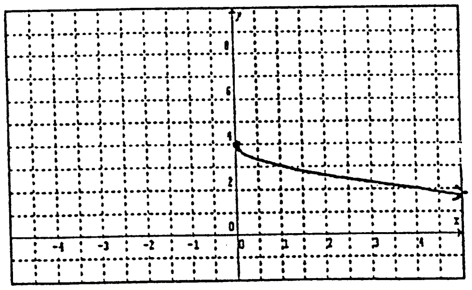
14. a)



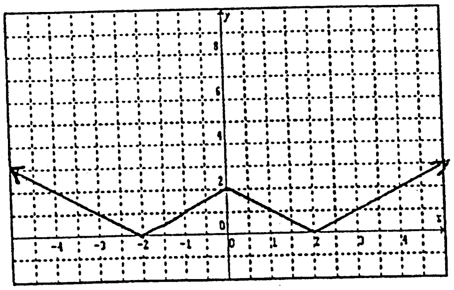
14 b)



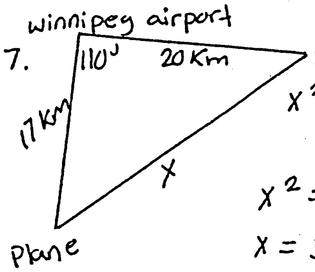
15.



16.



17.



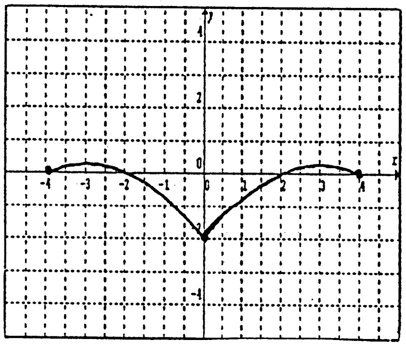
$x^2 = 17^2 + 20^2 - 2(17)(20)\cos 110°$

$x^2 = 921.5736975$

$x = 30.36$ Km

∴ The planes are 30.36 Km apart.

19.

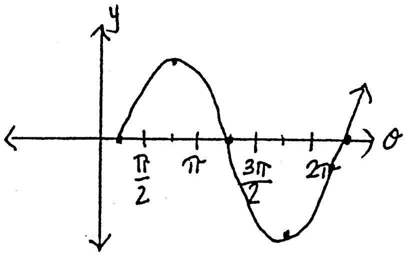


18. $\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

20.



Amplitude: 3
Period ; 2π

a) In terms of $\sin\theta$ ⇛ phase shift: $\pi/4$ to the right.

$\therefore y = 3\sin(\theta - \pi/4)$

b) In terms of $\cos\theta$ ⇛ phase shift: $3\pi/4$ to the right.

$\therefore y = 3\cos(\theta - 3\pi/4)$

Exercise 20: Solve Exponential Equations

1. $2^x = 32$

$2^x = 2^5$

$x = 5$

2. $2^{3x-5} = 16$

$2^{3x-5} = 2^4$

$3x - 5 = 4$

$3x = 9$

$x = 3$

3. $5^{4x-7} = 125$

$5^{4x-7} = 5^3$

$4x - 7 = 3$

$4x = 10$

$x = 5/2$

4. $3^{x^2+4x} = \frac{1}{27}$

$3^{x^2+4x} = 3^{-3}$

$x^2 + 4x = -3$

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

$x = -3$ or $x = -1$

5. $\frac{1}{3^{x-1}} = 81$

$3^{-(x-1)} = 3^4$

$-x + 1 = 4$

$-x = 3$

$x = -3$

6. $2(5^{2x-9}) = 250$

$5^{2x-9} = 125$

$5^{2x-9} = 5^3$

$2x - 9 = 3$

$2x = 12$

$x = 6$

7. $3^{8x} = \frac{1}{81}$

$3^{8x} = 3^{-4}$

$8x = -4$

$x = -0.5$

8. $32^{3x-2} = 16$

$(2^5)^{3x-2} = 2^4$

$5(3x-2) = 4$

$15x - 10 = 4$

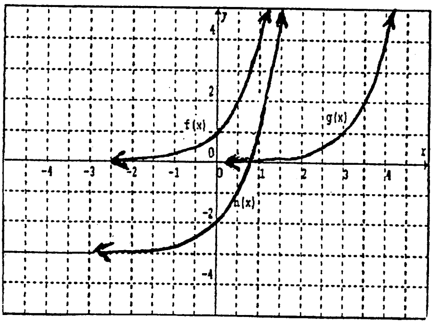
$15x = 14$

$x = 14/15$

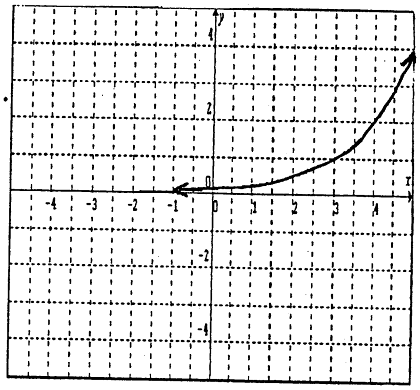
9. $y = 2^{-x} \Rightarrow y = \frac{1}{2^x} \Rightarrow y = \left(\frac{1}{2}\right)^x$

∴ (a) and (c) are equivalent

10.



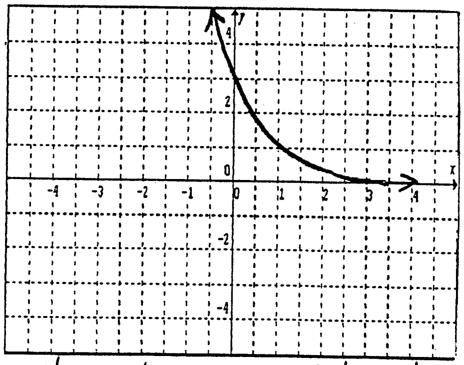
11.



x-intercepts: None

y-intercepts: 1/8

12



x-intercepts : None y-intercepts : 3

Horizontal Asymptote: $y=0$

Domain: $\{x \in R\}$ Range: $\{y \mid y>0\}$

13. LHS: $\frac{\sec^2 x}{\tan^2 x}$

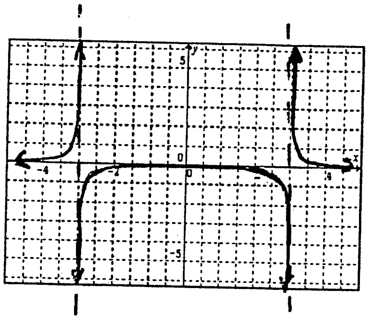
$$\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}$$

$$\frac{1}{\sin^2 x}$$

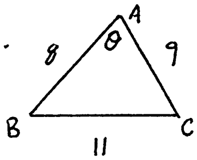
$= \csc^2 x = RHS$

Exercise 20: Solve Exponential Equations (continued)

14.



15.



$11^2 = 8^2 + 9^2 - 2(8)(9)\cos\theta$

$121 = 145 - 144\cos\theta$

$-24 = -144\cos\theta$

$\cos\theta = 0.1666667$

$\theta = 80.41°$

16. $\sin\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{5\pi}{6}\right) = \sin\theta\cos{}^{2\pi}/_{3} + \cos\theta\sin{}^{2\pi}/_{3} + \cos\theta\cos{}^{5\pi}/_{6}$

$- \sin\theta\sin{}^{5\pi}/_{6}$

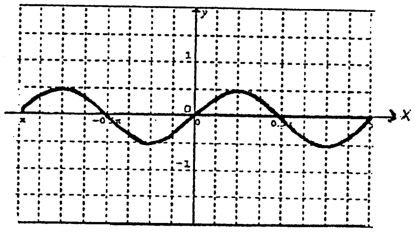
$= \sin\theta(-\tfrac{1}{2}) + \cos\theta(\tfrac{\sqrt{3}}{2}) + \cos\theta(\tfrac{-\sqrt{3}}{2}) - \sin\theta(\tfrac{1}{2})$

$= -\tfrac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta - \frac{\sqrt{3}}{2}\cos\theta - \tfrac{1}{2}\sin\theta$

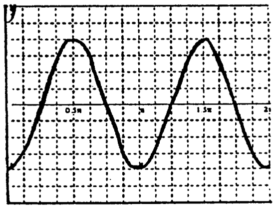
$= -1\sin\theta + 0\cos\theta$

$\therefore A = -1 \text{ and } B = 0$

17.



18.



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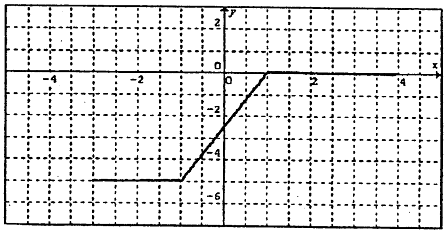
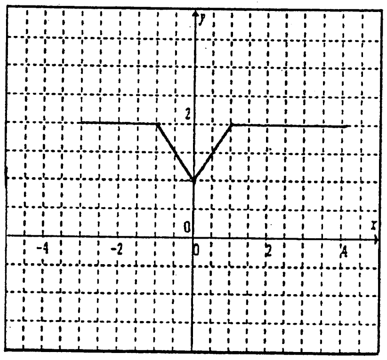


20.



Exercise 21: Logarithmic Functions

1. $b^x = y \Rightarrow x = \log_b y$ a) $3^4 = 81 \Rightarrow 4 = \log_3 81$

b) $16 = 2^4 \Rightarrow 4 = \log_2 16$ c) $\left(\frac{1}{4}\right)^2 = \frac{1}{16} \Rightarrow 2 = \log_{1/4}\left(\frac{1}{16}\right)$ d) $2^{-3} = \frac{1}{8} \Rightarrow -3 = \log_2\left(\frac{1}{8}\right)$

2. $\log_b x = y \Rightarrow b^y = x$

a) $\log_2 16 = 4 \Rightarrow 2^4 = 16$ b) $\log_4 64 = 3 \Rightarrow 4^3 = 64$

c) $\log_{10} 0.01 = -2 \Rightarrow 10^{-2} = 0.01$ d) $\log_5 \left(\frac{1}{5}\right) = -1 \Rightarrow 5^{-1} = \frac{1}{5}$

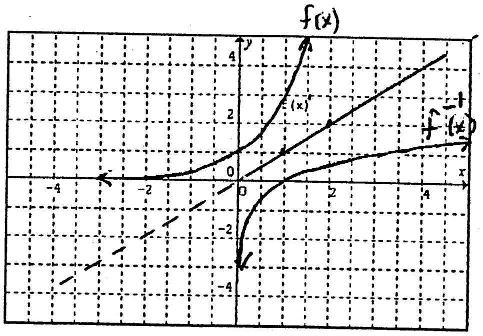
3. a) $\log_4 16 = x$
$4^x = 16$
$4^x = 4^2$
$x = 2$

b) $\log_9 3 = x$
$9^x = 3$
$(3^2)^x = 3^1$
$2x = 1$
$x = \frac{1}{2}$

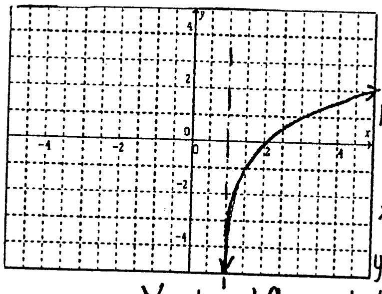
c) $\log_{\sqrt{2}} 8 = x$
$\sqrt{2}^x = 8$
$(2^{1/2})^x = 2^3$
$\frac{1}{2}x = 3$
$x = 6$

d) $\log_2(\log_3 9) = x$
$2^x = \log_3 9$
$\log_3 9 = y$
$3^y = 9$
$y = 2$
$\therefore 2^x = 2^1$
$x = 1$

4.

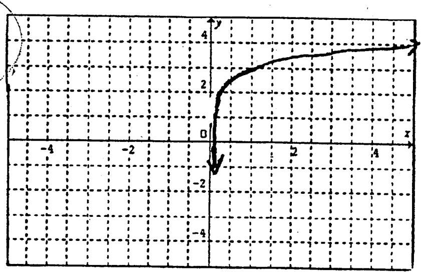


5.



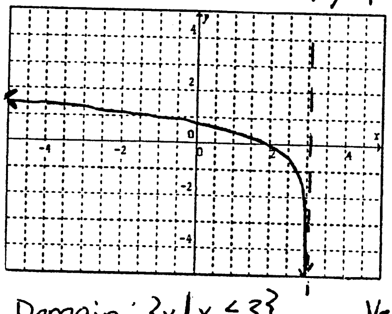
Domain: $\{x \mid x > 1\}$
Range: $\{y \in R\}$
x-intercept: 2
y-intercept: none
Vertical Asymptote: $x = 1$

6.



Domain: $\{x \mid x > 0\}$
Range: $\{y \in R\}$
x-int. $1/125$
y-int. none
Vertical Asymptote: $x = 0$

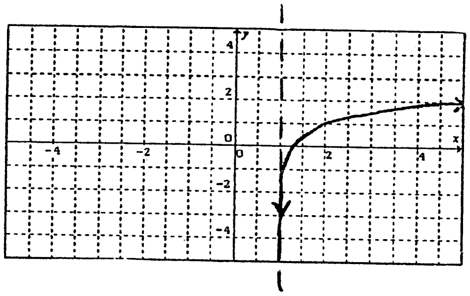
7.



Domain: $\{x \mid x < 3\}$
Range: $\{y \in R\}$
x-int: 2 y-int: 0.7925
Vertical Asymptote: $x = 3$

Exercise 21: Logarithmic Functions (continued)

8.



Domain : $\{x \mid x > 1\}$
Range : $\{y \in R\}$
x-int : $5/4$
y-int : None
Vertical Asymptote: $x = 1$

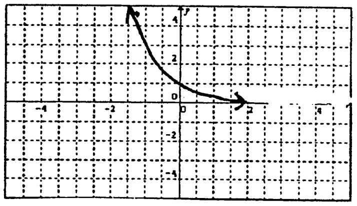
9. $2^{x^2} = 16$
$2^{x^2} = 2^4$
$x^2 = 4$
$x = \pm 2$

10. $8^{2x+1} = 64$
$8^{2x+1} = 8^2$
$2x + 1 = 2$
$2x = 1$
$x = \frac{1}{2}$

11. $\frac{1}{4^{x-2}} = 64$
$4^{-(x-2)} = 4^3$
$-x + 2 = 3$
$-x = 1$
$x = -1$

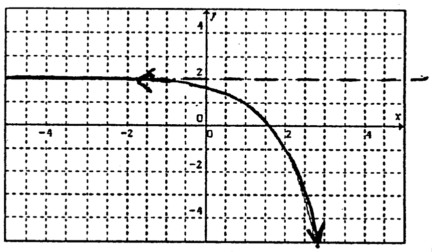
12. $\left(\frac{3}{5}\right)^x = \frac{27}{125}$
$\frac{3^x}{5^x} = \frac{3^3}{5^3}$
$\therefore x = 3$

13.



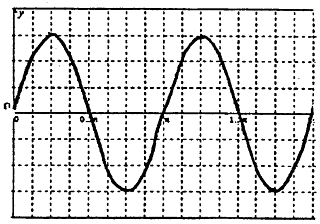
Asymptote : $y = 0$
x-int: none y-int: 1
Domain : $\{x \in R\}$
Range : $\{y \mid y > 0\}$

14.

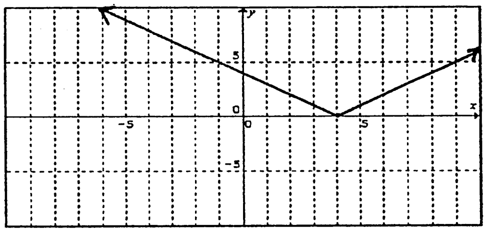


Domain : $\{x \in R\}$ Range : $\{y \mid y < 2\}$
Asymptote : $y = 2$
y-int : $5/3$

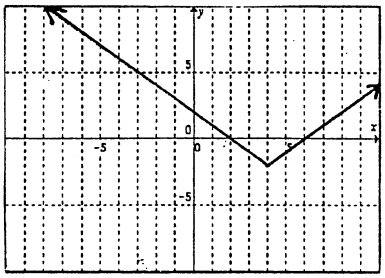
15.



16 .

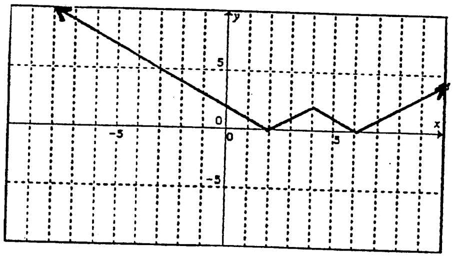


17.



Exercise 21: Logarithmic Functions (continued)

18.



19. $\sin\theta + 2\sin\theta\cos\theta = 0$

$\sin\theta(1+2\cos\theta)=0$

$\sin\theta = 0$ or $1+2\cos\theta = 0$

$\cos\theta = -1/2$

$\theta = 0, \pi, 2\pi$ $\theta = 2\pi/3, 4\pi/3$

20. $\cos\theta = -0.491$

$\theta_1 = 1.06$ - related angle

$\theta = 2.08, 4.20, 8.37, 10.48$

Exercise 22: Logarithmic Theorems I

1. a) $\log_4(2 \cdot 32) = \log_4 64 = 3$

b) $\log x^4 - \log (x^2+1)^{1/3} + \log (x-1)^2$

$= \log \left(\frac{x^4(x-1)^2}{(x^2+1)^{1/3}}\right) = \log \left(\frac{x^4(x-1)^2}{\sqrt[3]{x^2+1}}\right)$

2. a) $\log_{10} 8^{-1/3}$

$= \log_{10}(1/2)$

b) $\log x^3 - \log y^2 - \log t^4 + \log b^{1/2}$

$= \log\left(\frac{x^3 b^{1/2}}{y^2 t^4}\right) = \log\left(\frac{x^3\sqrt{b}}{y^2 t^4}\right)$

3. a) $\log_3(x\sqrt{y}) = \log_3(xy^{1/2})$

$= \log_3 x + \log_3 y^{1/2}$

$= \log_3 x + 1/2 \log_3 y$

b) $\log\left(\frac{x^3y^4}{z^6}\right) = \log x^3 + \log y^4 - \log z^6$

$= 3\log x + 4\log y - 6\log z$

c) $\log_2(6x) = \log_2 6 + \log_2 x$

d) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$

$= 3\log_5 x + 6\log_5 y$

4. Let $x = \log_a M \Rightarrow a^x = M$

$y = \log_a N \Rightarrow a^y = N$

$\therefore MN = a^x a^y$

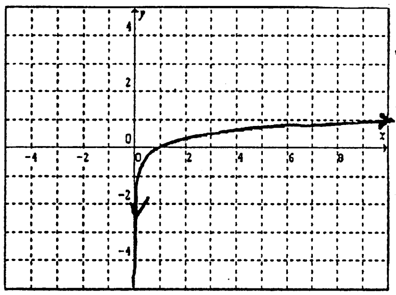
$MN = a^{x+y}$

$\log_a MN = \log_a(a^{x+y})$

$\log_a MN = x + y$

$\therefore \log_a MN = \log_a M + \log_a N$

5.



$y = \log x^2 - \log x$

$y = \log\left(\frac{x^2}{x}\right)$

$y = \log x$

6. $\log_3 \sqrt[5]{36} = \log_3 36^{1/5}$

$= \log_3 (6^2)^{1/5}$

$= \log_3 6^{2/5}$

$= \frac{2}{5} \log_3 6$

7. $\log_8 12 = \log_8(4 \cdot 3) = \log_8(2 \cdot 2 \cdot 3)$

$= \log_8 2 + \log_8 2 + \log_8 3$

$= 0.33333 + 0.33333 + 0.52832$

$= 1.19498$

$\log_8 36 = \log_8(4 \cdot 9) = \log_8(2 \cdot 2 \cdot 3 \cdot 3)$

$= \log_8 2 + \log_8 2 + \log_8 3 + \log_8 3$

$= 0.33333 + 0.33333 + 0.52832 + 0.52832$

$= 1.7233$

8. $\log\left(\frac{8a^3b^4}{5c^6}\right)$

$= \log 8a^3b^4 - \log 5c^6$

$= \log 8 + \log a^3 + \log b^4 - (\log 5 + \log c^6)$

$= \log 8 + 3\log a + 4\log b - \log 5 - 6\log c$

Exercise 22: Logarithmic Theorems I (continued)

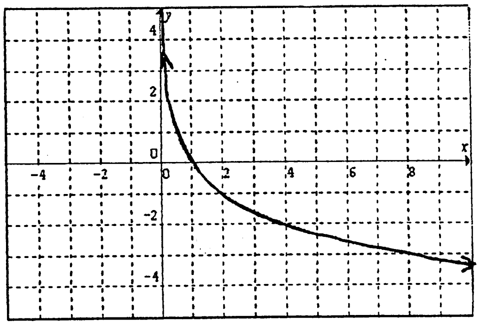
9. a) $6^4 = 1296 \Rightarrow 4 = \log_6 1296$

b) $5^{-3} = \frac{1}{125} \Rightarrow -3 = \log_5 \left(\frac{1}{125}\right)$

10. a) $\log_8 64 = 2 \Rightarrow 8^2 = 64$

b) $\log_4 \left(\frac{1}{64}\right) = -3 \Rightarrow 4^{-3} = \frac{1}{64}$

11.



Domain: $\{x \mid x > 0\}$
Range: $\{y \in R\}$
x-int : 1
y-int : none
Asymptote: $x = 0$

12. a) $\log_2 \left(\frac{1}{64}\right) = x$
$2^x = \frac{1}{64}$
$2^x = 2^{-6}$
$x = -6$

b) $\log_x 81 = 2$
$x^2 = 81$
$x = \pm 9$
But the base must be positive,
$\therefore x = 9$

13. $3^{9x} = \frac{1}{27}$
$3^{9x} = 3^{-3}$
$9x = -3$
$x = -\frac{1}{3}$

14. $\left(\frac{9}{25}\right)^x = \frac{5}{3}$
$\left(\frac{3^2}{5^2}\right)^x = \frac{5}{3}$
$\left(\frac{3}{5}\right)^{2x} = \left(\frac{3}{5}\right)^{-1}$
$2x = -1$
$x = -\frac{1}{2}$

15. $y = 3^{x-1}$
D: $\{x \mid x \in R\}$
R: $\{y \mid y > 0\}$
X-int : none
y-int: $y = \frac{1}{3}$

$y = \frac{1}{3}(3^x)$
D: $\{x \mid x \in R\}$
R: $\{y \mid y > 0\}$
x-intercept: none
y-intercept : $y = \frac{1}{3}$

These graphs are the same.
Note: $\frac{1}{3}(3^x) = 3^{-1}(3^x) = 3^{x-1}$

16. $LHS = \frac{\sec^2 x - \tan^2 x}{1 + \cot^2 x}$
$= \frac{1 + \tan^2 x - \tan^2 x}{\csc^2 x} = \frac{1}{\csc^2 x}$
$= \sin^2 x = RHS$

Exercise 22: Logarithmic Theorems I (continued)

17. $\tan^2\theta + \sec^2\theta = 3$

$\tan^2\theta + (1+\tan^2\theta) = 3$

$1 + 2\tan^2\theta = 3$

$2\tan^2\theta = 2$

$\tan^2\theta = 1$

$\tan\theta = 1$ or $\tan\theta = -1$

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

or

$(\sec^2\theta - 1) + \sec^2\theta = 3$

$2\sec^2\theta - 1 = 3$

$2\sec^2\theta = 4$

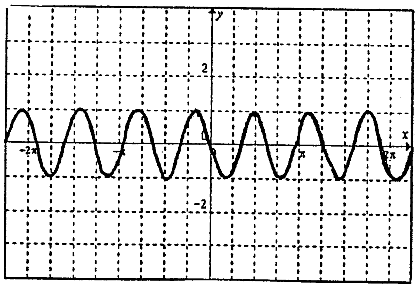
$\sec^2\theta = 2$

$\sec\theta = \pm\sqrt{2}$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

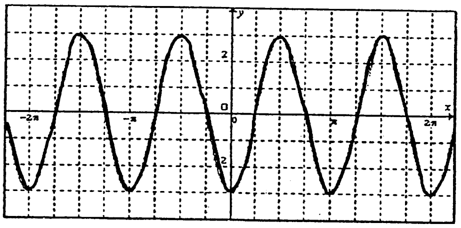
18. a)



b) Period = 2

(Period $= \frac{2\pi}{\pi} = 2$)

19. a)

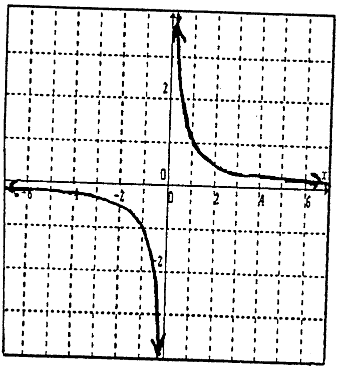


b) Period = π

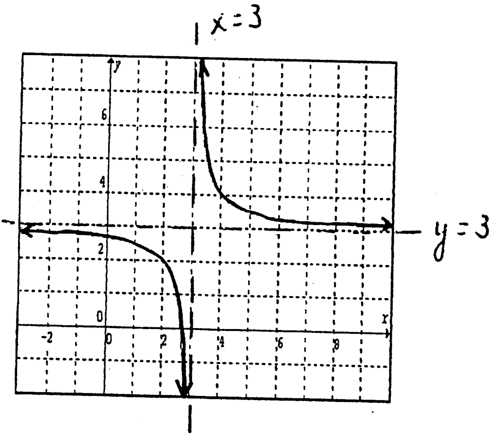
(Period $= \frac{2\pi}{2} = \pi$)

c) Horizontal shift $= \frac{-\pi}{2}$

20. a)



b)



Exercise 23: Logarithmic Theorems II

1. a) $\log_2 5+\log_2 7+\log_2 6=$

$\log_2(5\cdot7\cdot6)$

$=\log_2 210$

b) $\log_5 4+\log_5 6-\log_5 3$

$=\log_5\left(\frac{4\cdot6}{3}\right)$

$=\log_5 8$

2.a) $2\log_3 7-(\log_3 14+\log_3 35)$

$=\log_3 7^2-(\log_3(14\cdot35))$

$=\log_3\left(\frac{49}{14\cdot35}\right)$

$=\log_3\left(\frac{1}{10}\right)$

b) $\frac{1}{2}\log_2 4+\frac{1}{3}\log_2 27$

$=\log_2 4^{1/2}+\log_2 27^{1/3}$

$=\log_2(\sqrt{4}\sqrt[3]{27}=\log_2(2\cdot3)=\log_2 6$

c) $\log_b\left(\frac{14}{3}\right)=\log_b\left(\frac{2\cdot7}{3}\right)$

$=\log_b 2+\log_b 7-\log_b 3$

$=0.3010+0.8451-0.4771$

$=0.669$

d) $\log_b(\sqrt[3]{96})=\log_b 96^{1/3}=1/3\log_b 96$

$=1/3\log_b(2^5\cdot3)=1/3(\log_b 2^5+\log_b 3)$

$=1/3(5\log_b 2+\log_b 3)=1/3(5(0.3010)+0.4771)$

$=1/3(1.9821)=0.6607$

4. Let $y=\log_b x\Rightarrow b^y=x$

$\log_a b^y=\log_a x$

$y\log_a b=\log_a x$

$y=\frac{\log_a x}{\log_a b}$

$\therefore\log_b x=\frac{\log_a x}{\log_a b}$

5. False

$\log_4\left(\frac{a}{b}\right)=\log_a a-\log_a b$

6. $\log_8 7=\frac{\log 7}{\log 8}=0.935785$

7. $\log_a\left(\frac{3b\sqrt{c+1}}{4d^2}\right)$

$=\log_a\left(\frac{3b(c+1)^{1/2}}{4d^2}\right)$

$=\log_a 3+\log_a b+\log_a(c+1)^{1/2}-\log_a 4-\log_a d^2$

$=\log_a 3+\log_a b+\frac{1}{2}\log_a(c+1)-\log_a 4-2\log_a d$

8. $4\log_3 x-2\log_3 y+3\log_3 t-4\log_3 k$

$=\log_3 x^4-\log_3 y^2+\log_3 t^3-\log_3 k^4$

$=\log_3\left(\frac{x^4t^3}{y^2k^4}\right)$

Exercise 23: Logarithmic Theorems II (continued)

9. $25^{2x+1} = 125$

$(5^2)^{2x+1} = 5^3$

$5^{4x+2} = 5^3$

$4x+2 = 3$

$4x = 1$

$x = \frac{1}{4}$

10. $2^{2x+2} = \frac{1}{16}$

$2^{2x+2} = 2^{-4}$

$2x+2 = -4$

$2x = -6$

$x = -3$

11. $3^{-2} = 1/9$

$\log_3 (1/9) = -2$

12. $\log_2 32 = 5$

$2^5 = 32$

13. $\cos^2\theta + \sin^2\theta + 2\sin\theta = 4$

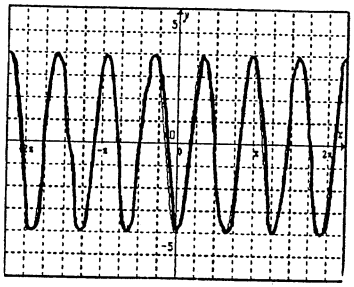
$1 + 2\sin\theta = 4$

$2\sin\theta = 3$

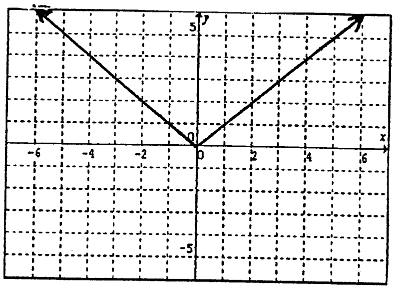
$\sin\theta = 3/2$

Since $\sin\theta > 1$, there is no solution.

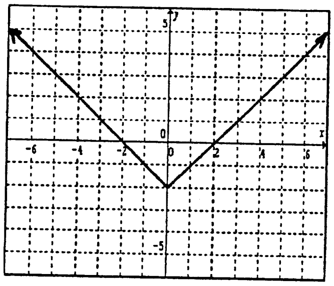
14.



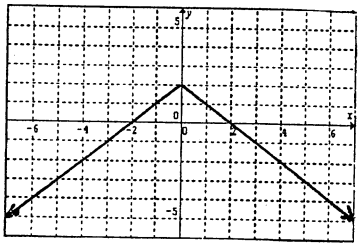
15. a)



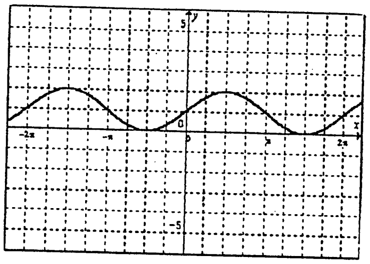
15 b)



c)

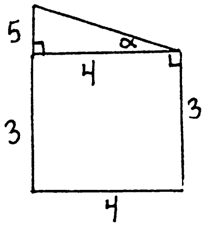


16.



Exercise 23: Logarithmic Theorems II (continued)

17.



$\tan \alpha = \frac{5}{4} \Rightarrow \alpha = 51.34^\circ$

$\therefore \theta = 90^\circ + 51.34^\circ = 141.34^\circ$

18. $f(x) = 2x+3$

$y = 2x+3$

$\therefore$ Inverse: $x = 2y+3$

$x-3 = 2y$

$\frac{x-3}{2} = y$

$\therefore f^{-1}(x) = \frac{x-3}{2}$

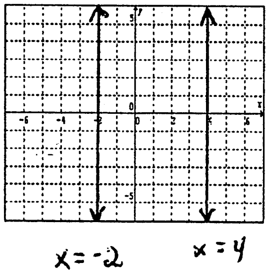
19. B

20. $(x-1)^2 = 9$

$x-1 = \pm 3$

$x-1 = 3$ or $x-1 = -3$

$x = 4$ or $x = -2$



Exercise 24: Exponential and Logarithmic Equations I

1\. a) $\log_5 x = 2$

$5^2 = x$

check: $\log_5 25 = 2$

$5^2 = 25$

True

b) $\log_x 25 = 2$

$x^2 = 25$

$x^2 = 5^2$

$x = 5$

check: $\log_5 25 = 2$

$5^2 = 125$

True

c) $\log_x 16 = -4/3$

$x^{-4/3} = 16$

$(x^{-4/3})^{-3/4} = (16)^{-3/4}$

$x = \frac{1}{16^{3/4}}$

$x = \frac{1}{2^3}$

$x = \frac{1}{8}$

check: $\log_{\frac{1}{8}} 16 = -4/3$

$(1/8)^{-4/3} = 16$

$(8)^{4/3} = 16$

$(2)^4 = 16$

$16 = 16$ True

2\. a) $\log_{1/2} (1/2)^4 = x$

$(1/2)^x = (1/2)^4$

$x = 4$

check: $\log_{\frac{1}{2}} (1/2)^4 = 4$

True

b) $\log_x x^4 = 4$

$x^4 = x^4$

answer $x > 0$ and $x \in$ Reals

c) $\log_7 1 = x$

$7^x = 1$

$x = 0$

check: $\log_7 1 = 0$

True

3\. a) $\log_x 16 = 4/3$

$x^{4/3} = 16$

$(x^{4/3})^{3/4} = (16)^{3/4}$

$x = (2)^3$

$x = 8$

check: $\log_8 16 = 4/3$

True

3\. b) $\log_2 x = 2$

$2^2 = x$

$4 = x$

check: $\log_2 4 = 2$

True

c) $\log_{16} 2 = x$

$16^x = 2$

$(2^4)^x = 2$

$4x = 1$

$x = 1/4$

check: $\log_{16} 2 = 1/4$

True

d) $\log_{32} x = 2/5$

$(32)^{2/5} = x$

$2^2 = x$

$4 = x$

check: $\log_{32} 4 = 2/5$

$\therefore$ b and d have the same solution.

Exercise 24: Exponential and Logarithmic Equations I (continued)

4. a) $\log_5 25^5 = x$

$5^x = 25^5$

$5^x = (5^2)^5$

$x = 10$

b) $27^{\log_3 9} = x$

since $\log_3 9 = 2$,

$27^2 = x$

$729 = x$

c) $\log_6(\log_2 64) = x$

since $\log_2 64 = 6$, $\log_6 6 = x$

$6^x = 6^1$

$x = 1$

5. a) $\log_x \sqrt{5} = \frac{1}{4}$

$x^{\frac{1}{4}} = \sqrt{5}$

$(x^{1/4})^{4/1} = (\sqrt{5})^4$

$x = 25$

check: $\log_{25} \sqrt{5} = \frac{1}{4}$ True

c) $5^{x-2} = 1$

$5^{x-2} = 5^0$

$x - 2 = 0$

$x = 2$

b) $\log_3 x - \log_3 4 = \log_3 12$

Method 1

$\log_3 x - \log_3 4 - \log_3 12 = 0$

$\log_3 \frac{x}{4(12)} = 0$

$\log_3 \frac{x}{48} = 0$

$3^0 = \frac{x}{48}$

$1 = \frac{x}{48}$

$x = 48$

check

or

Method 2

$\log_3 x = \log_3 4 + \log_3 12$

$\log_3 x = \log_3 48$

$x = 48$

check

6. $\log_5(x^2 - 4x) = 1$

$5^1 = x^2 - 4x$

$0 = x^2 - 4x - 5$

$0 = (x-5)(x+1)$

$x = 5, -1$

check: $\log_5(5^2 - 4 \cdot 5) = 1$

$\log_5 5 = 1$ True

$\log_5((-1)^2 - 4(-1)) = 1$

$\log_5(1+4) = 1$ $\log_5(5) = 1$

True

7. $\log_3 |3-2x| = 2$

$3^2 = |3-2x|$

$9 = |3-2x|$

$+9 = 3 - 2x$ or $-9 = 3 - 2x$

$6 = -2x$ $-12 = -2x$

$-3 = x$ $6 = x$

check:

$\log_3 |3-2(-3)| = 2$

$\log_3 |9| = 2$

True

$\log_3 |3-2(6)| = 2$

$\log_3 |-9| = 2$

True

Exercise 24: Exponential and Logarithmic Equations I (continued)

8. $\log(x+21)+\log x = 2$
$\log(x+21)x = 2$
$10^2 = (x+21)x$
$100 = x^2+21x$
$0 = x^2+21x-100$
$0 = (x+25)(x-4)$
$x = -25 \quad / x = 4$
$\downarrow$
check: extraneous root

9. $f(x) = -2^{-(x+2)}$
$g(x) = (2^2)^{x-2}$
$= 2^{2x-4}$
$h(x) = 2^{2x-4}$
$\therefore g(x) = h(x)$

10. $4^{6x} = \frac{1}{64}$
$(2^2)^{6x} = 2^{-6}$
$12x = -6$
$x = -\frac{1}{2}$

11.

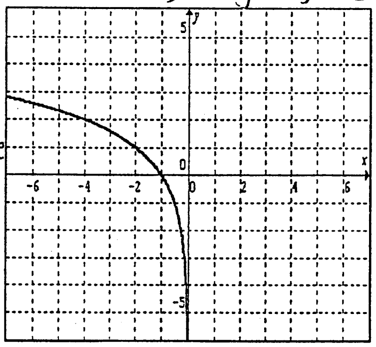
S	A ✓
✓T	C ✓

cot θ will be in Quad IV

$\cos^2\theta + \sin^2\theta = 1$
$(3/5)^2 + \sin^2\theta = 1$
$9/25 + \sin^2\theta = 1$
$\sin^2\theta = 1 - 9/25$
$\sin^2\theta = 16/25$
$\sin\theta = \pm 4/5$, since $\sin\theta < 0 \Rightarrow \sin\theta = \frac{-4}{5}$
$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{3/5}{-4/5} = -3/4$

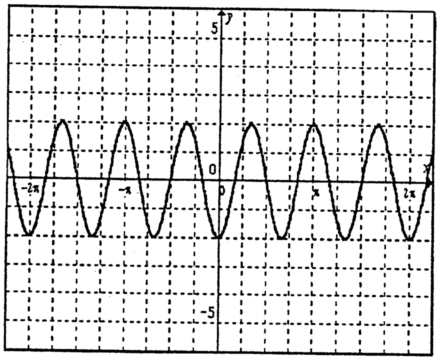
12. Domain $\{x < 0\}$ Range: $\{y \in R\}$

x-int: -1
y-int: none
asymptote y=0



13. $\frac{1}{\tan x} + \tan x = \sec x \csc x$
$L.H.S. = \frac{1}{\tan x} + \tan x$
$= \frac{1+\tan^2 x}{\tan x}$
$= \frac{\sec^2 x}{\tan x}$
$= \frac{\sec x \cdot \left(\frac{1}{\cos x}\right)}{\tan x}$
$= \frac{\sec x}{1} \cdot \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x}$
$= \sec x \cdot \frac{1}{\sin x}$
$= \sec x \csc x$
$= R.H.S$

14.



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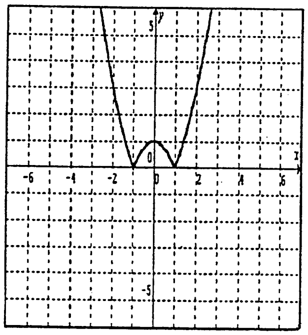
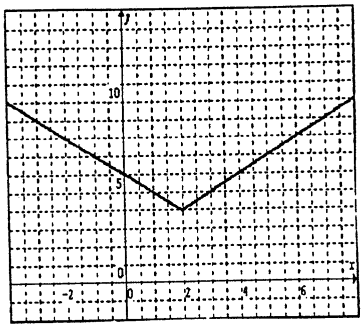


Exercise 24: Exponential and Logarithmic Equations I (continued)

16.



17. $x^2+4x+4 = x^2-2x+1+x^2$

solution 1:

$0 = x^2-6x-3$

$12 = x^2-6x-9$

$12 = (x-3)^2$

$x-3 = \pm 2\sqrt{3}$

$x = 3 \pm 2\sqrt{3}$

solution 2:

$x = \frac{6 \pm \sqrt{36-4(1)(3)}}{2(1)}$

$x = \frac{6 \pm \sqrt{48}}{2}$

$x = \frac{6 \pm 4\sqrt{3}}{2}$

$x = 3 \pm 2\sqrt{3}$

18. Amplitude: 3

Period: $\pi \Rightarrow b = 2$

horizontal shift:

cosine: $\pi/4$ left

sine: $\pi/2$ left

vertical shift: 0

$y = 3\cos(2(x+\pi/4))$

or

$y = 3\sin(2(x+\pi/2))$

other answers possible

solution 1:	Solution 2:
19. $y = \frac{3}{x-2}$	$y = \frac{3}{x-2}$
$x = \frac{3}{y-2}$	$x = \frac{3}{y-2}$
$\frac{1}{x} = \frac{y-2}{3}$	$xy - 2y = 3$
$\frac{3}{x} = y-2$	$xy = 3+2x$
$y^{-1} = 2 + \frac{3}{x}$	$y = \frac{3+2x}{x}$
	$y^{-1} = \frac{3+2x}{x}$

LHS.

20. $\log\left(\frac{\sqrt{3}}{2}\right)$

$= \log\sqrt{\frac{3}{4}}$

$= \frac{1}{2}\left[\log 3 - \log 4\right]$

Exercise 25: Exponential and Logarithmic Equations II

1. $\log_2(x-4)+\log_2(x-3)=1$

$\log_2(x-4)(x-3)=1$

$2^1=(x-4)(x-3)$

$2=x^2-7x+12$

$0=x^2-7x+10$

$0=(x-5)(x-2)$

$x=5$ check ✓ | $x=2$ extraneous root

2. $\log_4(x+2)-\log_4(2x-3)=0$

$\log_4\left(\frac{x+2}{2x-3}\right)=0$

$4^0=\frac{x+2}{2x-3}$

$1=\frac{x+2}{2x-3}$

$2x-3=x+2$

$x=5$ check

3. $\log_2 x+\log_2(x-2)=\log_2(9-2x)$

$\log_2 x+\log_2(x-2)-\log_2(9-2x)=0$

$\log_2\frac{x(x-2)}{9-2x}=0$

$2^0=\frac{x(x-2)}{9-2x}$

$1=\frac{x^2-2x}{9-2x}$

$9-2x=x^2-2x$

$0=x^2-9$

$0=(x-3)(x+3)$

$x=3$ ✓ check | $x=-3$ extraneous root

4. $2\log_2 x-\log_2(x-1)=2$

$\log_2 x^2-\log_2(x-1)=2$

$\log_2\frac{x^2}{x-1}=2$

$2^2=\frac{x^2}{x-1} = 4=\frac{x^2}{x-1}$

$4x-4=x^2$

$0=x^2-4x+4$

$0=(x-2)(x-2)$

$x=2$ check ✓

5. $\log_{1/7}x+\log_{1/7}(5x-28)=-2$

$\log_{1/7}x(5x-28)=-2$

$(1/7)^{-2}=x(5x-28)$

$49=5x^2-28x$

$0=5x^2-28x-49$

$0=(5x+7)(x-7)$

$x=-7/5\ x$ extraneous root | $x=7$ check

6. $4^x=15 \quad \log 4^x=\log 15$

$x\log 4=\log 15$

$x=\frac{\log 15}{\log 4}\Rightarrow x=\frac{1.1760913}{0.60206}$

$x\doteq 1.9534453$

$x\doteq 1.95$

Exercise 25: Exponential and Logarithmic Equations II (continued)

7. $5^{2x-3} = 8$

$\log 5^{2x-3} = \log 8$

$\dfrac{(2x-3)\log 5}{\log 5} = \dfrac{\log 8}{\log 5}$

$2x - 3 = \dfrac{0.90309}{0.69897}$

$2x-3 = 1.2920297$

$2x = 4.2920297$

$x = 2.1460149$

$x = 2.15$

or $\log 5^{2x-3} = \log 8$

$(2x-3) \log 5 = \log 8$

$2x \log 5 - 3 \log 5 = \log 8$

$2x \log 5 = \log 8 + 3 \log 5$

$x = \dfrac{\log 8 + 3 \log 5}{2 \log 5}$

$x = 2.15$

8. $6^{3x} = 2^{2x-3}$

$\log 6^{3x} = \log 2^{2x-3}$

$\dfrac{3x \log 6}{\log 6} = \dfrac{(2x-3)\log 2}{\log 6}$

$3x = \dfrac{(2x-3)(0.30103)}{0.7781513}$

$3x = (2x-3)(0.386)$

$3x = .772x - 1.158$

$2.228x = -1.158$

$x = -0.5197487$

$x = -0.52$

or $\log 6^{3x} = \log 2^{2x-3}$

$3x \log 6 = (2x-3) \log 2$

$3x \log 6 = 2x \log 2 - 3 \log 2$

$3x \log 6 - 2x \log 2 = -3 \log 2$

$x(3 \log 6 - 2 \log 2) = -\log 8$

$x\left(\log \frac{216}{4}\right) = -0.90309$

$x \log 54 = -0.90309$

$x = -\dfrac{0.90309}{\log 54}$

$x = -\dfrac{0.90309}{1.7323938} = -0.52$

9. $3^x = 2 \cdot 5^{2-x}$

$x \log 3 = \log 2 + (2-x) \log 5$

$x(\log 3 + \log 5) = \log 2 + 2 \log 5$

$x = \dfrac{\log 2 + 2 \log 5}{\log 3 + \log 5} = 1.445$

10. $\log \frac{1}{5} = \log 1 - \log 5$

$= 0 - \log 5$

$= -\log 5$

11. $\log_x 10\sqrt{10} = \frac{3}{2}$

$x^{3/2} = 10\sqrt{10}$

$x^{3/2} = 10(10^{1/2})$

$x^{3/2} = 10^{3/2}$

$x = 10$

Exercise 25: Exponential and Logarithmic Equations II (continued)

12. $\log_{1/9} 27 = x$

$(1/9)^x = 27$

$(3^{-2})^x = 3^3$

$-2x = 3$

$x = -3/2$

13. $\log_{10} .0001 = x$

$10^x = 0.0001$

$10^x = 10^{-4}$

$x = -4$

14. $\log_2 6 + \log_2 9 = \log_2 x$

$\log_2 (6 \cdot 9) = \log_2 x$

$\log_2 54 = \log_2 x$

$54 = x$

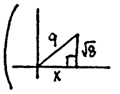
15. $10^{3x-1} = 10^{-4}$

$3x - 1 = -4$

$3x = -3$

$x = -1$

16. $\sin\theta = \frac{\sqrt{8}}{9}$ in QI $\Rightarrow$ $\cos\theta = \frac{\sqrt{73}}{9}$



$x^2 + (\sqrt{8})^2 = 9^2$

$x = \sqrt{73}$

$P(2\theta) = (\cos 2\theta, \sin 2\theta)$

$\cos 2\theta = 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{\sqrt{8}}{9}\right)^2$

$= \frac{65}{81}$

$\sin 2\theta = 2\sin\theta\cos\theta$

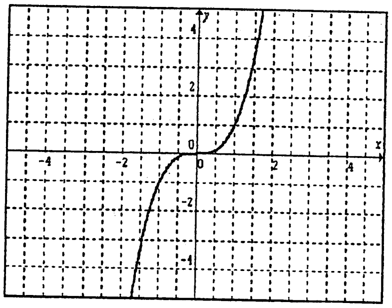
$= 2\left(\frac{\sqrt{8}}{9}\right)\left(\frac{\sqrt{73}}{9}\right)$

$= \frac{2\sqrt{584}}{81}$

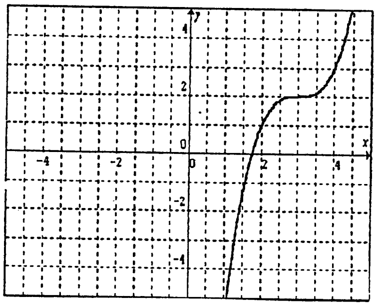
$= \frac{4\sqrt{146}}{81}$

$\therefore P(2\theta) = \left(\frac{65}{81}, \frac{4\sqrt{146}}{81}\right)$

17.



18.



19. $(5.5)\left(\frac{180}{\pi}\right)$

$= 315.13°$

20. $\sin\theta = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{4\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right)$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\cos\left(\frac{5\pi}{6}\right), \cos\left(\frac{7\pi}{6}\right)$

$\tan\theta = -\frac{\sqrt{3}}{2}$

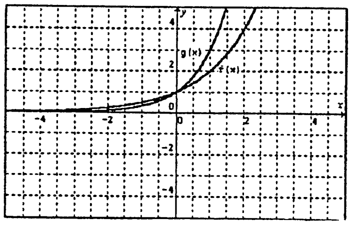
$\tan(2.428)$

$\tan(5.570)$

Exercise 26: Natural Logarithms

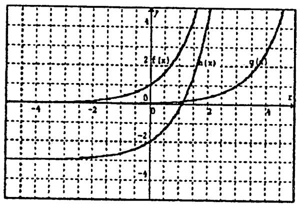
1. a) 221.41 b) 0.512 c) 2.244 d) −5.627

2.



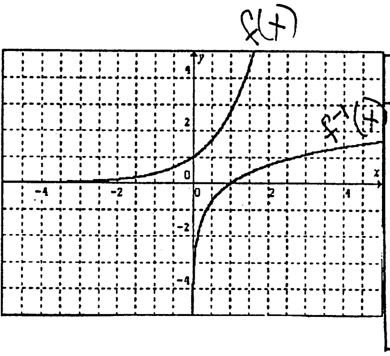
The graph of $h(x)$ is between $f(x)$ and $g(x)$ since $2 < e < 3$

3.



Property	f(x)	g(x)	h(x)
Domain	$\{x \mid x \in R\}$	$\{x / x \in R\}$	$\{x \mid x \in R\}$
Range	$\{y \mid y > 0\}$	$\{y \mid y > 0\}$	$\{y \mid y > -3\}$
x-intercept	none	none	$\ln 3 \approx 1.1$
y-intercept	1	≈ 0.05	-2
Asymptote(s)	$y = 0$	$y = 0$	$y = -3$

4.



Property	f(x)	$f^{-1}(x)$
Domain	$\{x \mid x \in R\}$	$\{x \mid x > 0\}$
Range	$\{y \mid y > 0\}$	$\{y \mid y \in R\}$
x-intercept	none	1
y-intercept	1	none
Asymptote(s)	$y = 0$	$x = 0$

5. a)
$$\begin{aligned}\ln\sqrt{x^3(x+1)} &= \ln\left(x^3(x+1)\right)^{1/2}\\ &= \tfrac{1}{2}\left[\ln x^3 + \ln(x+1)\right]\\ &= \tfrac{1}{2}\left[3\ln x + \ln(x+1)\right]\end{aligned}$$

b)
$$\begin{aligned}\ln\frac{(x-1)(x+3)^2}{\sqrt{x^2+2}} &= \ln(x-1) + \ln(x+3)^2 - \ln(x^2+2)^{1/2}\\ &= \ln(x-1) + 2\ln(x+3) - \tfrac{1}{2}\ln(x^2+2)\end{aligned}$$

Exercise 26: Natural Logarithms (continued)

6. a) $e^{-0.01x} = 27 \Rightarrow \ln e^{-0.01x} = \ln 27$

$-0.01x \ln e = \ln 27$

$-0.01x = \ln 27$

$x = -329.58369$

b) $e^{\ln(1-x)} = 2x$

$1 - x = 2x$

$1 = 3x$

$1/3 = x$

c) $\ln e^{\sqrt{x+1}} = 3$

$\sqrt{x+1}\ \ln e = 3$

$\sqrt{x+1} = 3$

$x + 1 = 9$

$x = 8$

d) $e^{2x-1} = 5$

$\ln e^{2x-1} = \ln 5$

$(2x-1) \ln e = \ln 5$

$2x - 1 = 1.6094379$

$2x = 2.60944$

$x = 1.30472$

7. a) $y = 80e^{-0.2x}$

$= 80e^{-0.2(3)}$

$= 80e^{-0.6}$

$= 80(0.5488)$

$= 43.90$ grams

after 3 years

b) $y = 80e^{-0.2x}$

$40 = 80e^{-0.2x}$

$1/2 = e^{-0.2x}$

$\ln 1/2 = \ln e^{-0.2x}$

$\ln 1 - \ln 2 = -0.2x \ln e$

$-\ln 2 = -0.2x$ $x = 3.466$ years

8. $A = Pe^{rt}$

$2000 = 1000e^{0.10t}$

$2 = e^{0.10t}$

$\ln 2 = \ln e^{0.10t}$

$0.6931472 = 0.10t$

$t = 6.93$ years

9. $5^{3n+1} = 5^4$

$3n + 1 = 4$

$3n = 3$

$n = 1$

10. $\log_2 16\sqrt{2} = \frac{9}{2}$

11. One possible solution:

L.H.S. $= \dfrac{1 - \cos^2\theta}{\sec^2\theta - 1}$

$= \dfrac{\sin^2\theta}{\tan^2\theta}$

$= \left(\dfrac{\sin^2\theta}{1}\right) \cdot \left(\dfrac{\cos^2\theta}{\sin^2\theta}\right)$

$= \cos^2\theta$ = R.H.S.

12. $\log_2 8\sqrt{2} = x$

$2^x = 8\sqrt{2}$

$2^x = 2^3(2^{1/2})$

$2^x = 2^{7/2}$

$x = 7/2$

13. $\log_{5n} 25n^2 = x$

$(5n)^x = 25n^2$

$(5n)^x = (5n)^2$

$x = 2$

Exercise 26: Natural Logarithms (continued)

14. $\log_5(3x+1) + \log_5(x-3) = 3$

$\log_5(3x+1)(x-3) = 3$

$5^3 = (3x+1)(x-3)$

$125 = 3x^2 - 8x - 3$

$0 = 3x^2 - 8x - 128$

$0 = (3x+16)(x-8)$

$x = -16/3x$ $x = 8$ ✓ check

reject

15. $\log(x^3-1) - \log(x^2+x+1) = 1$

$\log\left(\frac{x^3-1}{x^2-x+1}\right) = 1$

$\log\left(\frac{(x-1)(\cancel{x^2+x+1})}{\cancel{x^2+x+1}}\right) = 1$

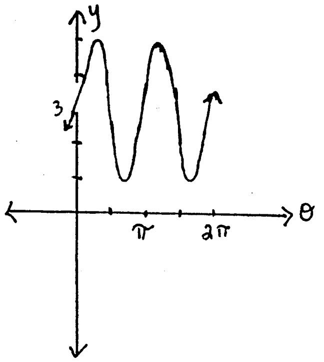
$\log(x-1) = 1$

$10^1 = x-1 \rightarrow 11 = x$ check

16. $\log\left(\frac{4(x-5)}{x^3(x+6)}\right) = \log 4(x-5) - \log x^3(x+6)$

$= \log 4 + \log(x-5) - 3\log x - \log(x+6)$

17. $y = 3 + 2\sin 2\theta$



18. $\frac{5^x}{7^{1-x}} = 6^{2-x}$

$x\ln 5 - (1-x)\ln 7 = (2-x)\ln 6$

$x\ln 5 - \ln 7 + x\ln 7$

$= 2\ln 6 - x\ln 6$

$x(\ln 5 + \ln 7 + \ln 6) = 2\ln 6 + \ln 7$

$5.3471x = 5.5294$

$x = 1.034$

19. One solution is to consider that a maximum of $y = \sin x$ occurs at $(\pi/2, 1)$. Therefore we could shift the function $\pi/2$ units to the left and 1 unit down to translate this max to $(0,0)$ Thus $A = 1$, $B = \pi/2$, $C = -1$.

20. $\sin\theta\cos\pi/4 + \cos\theta\sin\pi/4 + \cos\theta\cos 7\pi/6 + \sin\theta\sin 7\pi/6$

$= (\cos\pi/4 + \sin 7\pi/6)\sin\theta + (\sin\pi/4 + \cos 7\pi/6)\cos\theta$

$= \left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)\sin\theta + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\right)\cos\theta$ $\therefore P = \frac{\sqrt{2}-1}{2}$, $Q = \frac{\sqrt{2}-\sqrt{3}}{2}$

Exercise 27: Applications of the Exponential Function

1. a) $\ln\left[\frac{\sqrt{x}}{(x-1)^2\sqrt[3]{x^2+1}}\right]$

b) $\ln\left[\frac{x^3-1}{x^2+x+1}\right] = \ln\left[\frac{(x-1)(x^2+x+1)}{x^2+x+1}\right]$

$= \ln(x-1)$

2. a)

$5000 = 50e^{2K}$

$100 = e^{2K}$

$\ln 100 = \ln e^{2K}$

$\ln 100 = 2K$

$K = \frac{\ln 100}{2}$

b)

$\frac{A}{3} = Ae^{4K}$

$\frac{1}{3} = e^{4K}$

$\ln \frac{1}{3} = \ln e^{4K}$

$\ln 1 - \ln 3 = 4K$

$-\ln 3 = 4K \quad = -\frac{\ln 3}{4} = k$

3. $A = P\left(1+\frac{r}{n}\right)^{nt}$

a) $A = 5000\left(1+\frac{0.084}{12}\right)^{12(1)}$

$= 5000(1.007)^{12}$

$A = \$5436.55$ after 1 year

b) $A = 5000\left(1+\frac{0.084}{12}\right)^{12(10)}$

$= 5000(1.007)^{120}$

$A = \$11\,547.99$ after 10 years

c) $(11\,547.99 - 5000 = \$6\,547.99)$

Interest earned after 10 years is $6 547.99

d)

$20\,000 = P\left(1+\frac{0.084}{12}\right)^{12(5)}$

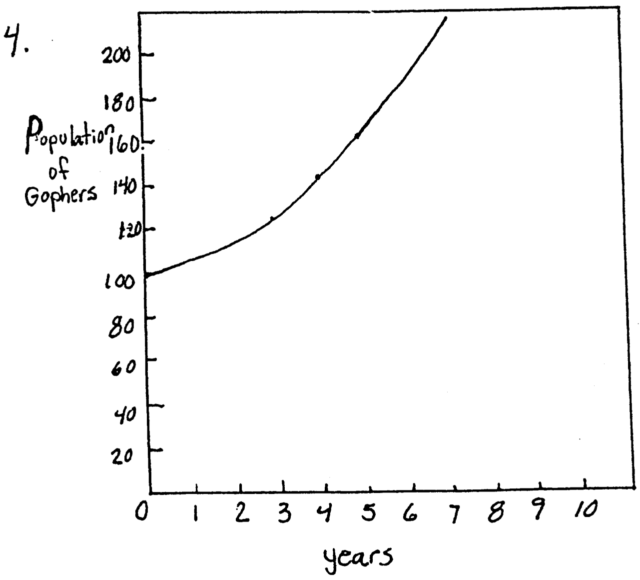
$20\,000 = P(1.007)^{60}$

$20\,000 = P(1.5197363)$

$\frac{20\,000}{1.5197363} = P$

P = $13160.18 is the initial investment.

4.



After 20 years, there will be 673 gophers; population will double after 7.3 years.

Exercise 27: Applications of the Exponential Function (continued)

5. $A = Pe^{rt}$

$6000 = 2000e^{(0.8)t}$

$3 = e^{0.08T}$

$\ln 3 = 0.08T$

$T = 13.73$ years

6. a) $y = Ae^{Kx}$

$8 = 10e^{5k}$

$.8 = e^{5K}$

$\ln .8 = 5K$

$K = \frac{\ln(0.8)}{5}$

b) $y = 10e^{\left(\frac{\ln .8}{5}\right)10}$

$y = 10e^{--0.4962871}$

$y = 6.4$ grams

6. c) $5 = 10e^{\left(\frac{\ln .8}{5}\right)x}$

$\frac{1}{2} = e^{-.0446287x}$

$\ln 1 - \ln 2 = .0446287x$

$\frac{-\ln 2}{-.0446287} = x \Rightarrow x = 15.5$ years

7. $P = Ae^{Kt}$

$\therefore 24000 = 22000e^{5K}$

$\ln \frac{24}{22} = 5K$

$K = 0.0174$

$\therefore P = Ae^{0.0174t}$

$44000 = 22000e^{0.0174t}$

$\ln 2 = 0.0174t$

$t = \frac{\ln 2}{0.0174}$

$= 39.83 \approx 40$ yrs

8. $ph = -\log[H+]$

$6.62 = -\log[H+]$

$10^{6.62} = H^{-1}$

$4168693.80 = H^{-1}$

$H = 2.0 \times 10^{-7}$

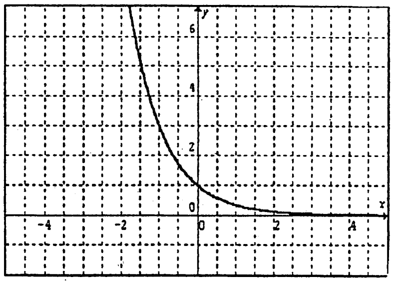
or $ph = -\log[H+]$

$6.62 = -\log[H+]$

$-6.62 = \log[H+]$

$2.0 \times 10^{-7} = H+$

9.



b) x - Intercept

$0 = 3^{-x}$

There is no x-intercept.

y - Intercept

$y = 3^{-0}$

$y = 1$

10. $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x} = 1$

proof: LHS: $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$

$= \frac{\sin x}{\frac{1}{\sin x}} + \frac{\cos x}{\frac{1}{\cos x}}$

$= \sin^2 x + \cos^2 x$

$= 1 = \text{RHS}$

11. $\log_x \frac{1}{9} = -2$

$x^{-2} = \frac{1}{9}$

$(x^{-2})^{-\frac{1}{2}} = (\frac{1}{9})^{-\frac{1}{2}}$

$x = \frac{1}{(\frac{1}{9})^{\frac{1}{2}}}$

$x = \frac{1}{\frac{1}{3}}$

$x = 3$

12. $\log_{\frac{3}{5}} \frac{27}{125} = x$

$\left(\frac{3}{5}\right)^x = \frac{3^3}{5^3}$

$\left(\frac{3}{5}\right)^x = \left(\frac{3}{5}\right)^3$

$x = 3$

Exercise 27: Applications of the Exponential Function (continued)

13. a) $\log 5 + x \log 23 = (1-x) \log 47$

$\log 5 + x \log 23 = \log 47 - x \log 47$

$x(\log 23 + \log 47) = (\log 47 - \log 5)$

$3.0338\ x = 0.9731$

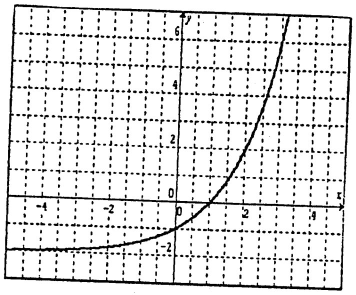
$x = 0.321$

b) $x(\ln 23 + \ln 47) = (\ln 47 - \ln 5)$

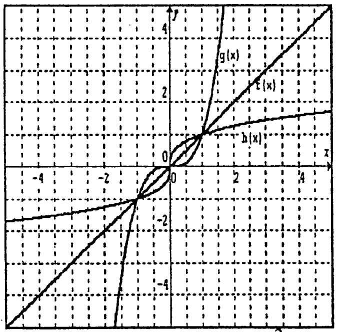
$6.9856\ x = 2.5407$

$x = 0.321$

14.



15.



Note: $f(x) = x$; $g(x) = x^3$; $h(x) = x^{1/3}$

The graph of $y = x^3$ is reflected in the $y = x$ line to give the graph of $y = \sqrt[3]{x}$.

16. $f(x) = x^3$

$y = x^3$

Inverse: $x = y^3$

or $y = \sqrt[3]{x}$

$\therefore f^{-1}(x) = \sqrt[3]{x}$

17. $5 \sin\theta - 12 \cos^2\theta + 10 = 0$

$5 \sin\theta - 12(1 - \sin^2\theta) + 10 = 0$

$12 \sin^2\theta + 5 \sin\theta - 2 = 0$

$(3 \sin\theta + 2)(4 \sin\theta - 1) = 0$

$\therefore \sin\theta = -2/3$ or $\sin\theta = 1/4$

$\theta_R = 0.729$ $\theta_R = 0.2527$

$\theta_1 = 3.871$ $\theta_3 = .2527$

$\theta_2 = 5.554$ $\theta_4 = 2.889$

$\therefore \{3.871, 5.554, 0.2527, 2.889\}$

18. $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{8}{10} = \frac{4}{5}$

Exercise 27: Applications of the Exponential Function (continued)

19. a) $\tan(x+\pi) = \dfrac{\tan x + \tan \pi}{1 - \tan x \tan \pi}$

$= \dfrac{\tan x + 0}{1 - \tan x (0)} = \tan x$

b) $\tan(x-\pi) = \dfrac{\tan x - \tan \pi}{1 + \tan x \tan \pi}$

$= \dfrac{\tan x - 0}{1 + \tan x (0)} = \tan x$

c) $\cot(\pi/2 - x) = \dfrac{\cot(\frac{\pi}{2})\cot x + 1}{\cot x - \cot \pi/2}$

$= \dfrac{0(\cot x) + 1}{\cot x - 0}$

$= \dfrac{1}{\cot x}$

$= \tan x$

d) $\cot(\pi/2 + x) = \dfrac{\cot \pi/2 \cot x - 1}{\cot x + \cot \pi/2}$

$= \dfrac{-1}{\cot x}$

$= -\tan x$

$\therefore$ D is not equal to $\tan x$.

20. $-15x = 7x^2 - 2x^3$

$2x^3 - 7x^2 - 15x = 0$

$x(2x^2 - 7x - 15) = 0$

$x(2x+3)(x-5) = 0$

$x = 0, -3/2, 5$

Exercise 28: Counting Principles

1\. a) $\frac{7!}{6!} = \frac{7\cdot\cancel{6!}}{\cancel{6!}} = 7$ b) $\frac{(31)!}{(30)!} = \frac{(31)(30)!}{(30)!} = 31$

c) $\frac{9!}{6!} = \frac{9\cdot8\cdot7\cdot6!}{6!} = 504$ d) $\frac{10!}{6!\,4!} = \frac{10\cdot9\cdot8\cdot7\cdot\cancel{6!}}{\cancel{6!}\,4\cdot3\cdot2\cdot1} = 210$

2\. a) $\frac{(K+3)!}{(K+2)!} = \frac{(K+3)(K+2)!}{(K+2)!} = K+3$

b) $\frac{7!(r+2)!r}{6!(r-1)!} = \frac{7\cdot\cancel{6!}(r+2)(r+1)(r)\cancel{(r-1)!}r}{\cancel{6!}\cancel{(r-1)!}} = 7(r+2)(r+1)r^2$

3\. $\frac{n!}{(n-2)!} = 20$

$n(n-1) = 20$

$n^2 - n - 20 = 0$

$(n-5)(n+4) = 0$

$n = 5$ (note: $n \neq -5$ since $n \geq 2$)

4\. i) $\frac{n!}{(n+1)(n!)} = \frac{1}{n+1}$

ii) $\frac{n(n-1)!}{(n-1)!} = n$

iii) $\frac{(n+1)(n!)n}{n!(n+1)} = n$

iv) $\frac{(n+1)n!}{n!} = n+1$

v) $\frac{n^2(n-1)!}{n(n-1)!} = n$

$\therefore$ ii, iii, v

5\. $\underset{\text{nickel}}{\underline{2}} \cdot \underset{\text{dime}}{\underline{2}} = 4$ ways

6\. $\underset{\text{1st}}{\underline{12}} \cdot \underset{\text{2nd}}{\underline{11}} \cdot \underset{\text{3rd}}{\underline{10}} = 1320$ ways

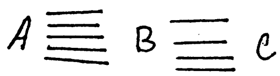
7\. $\underset{\text{salad}}{\underline{3}} \cdot \underset{\text{pizza}}{\underline{20}} \cdot \underset{\text{dessert}}{\underline{4}} = 240$ meals

8\. $\underset{\text{mod. lang}}{\underline{4}} \cdot \underset{\text{not sc.}}{\underline{5}} \cdot \underset{\text{soc. sc.}}{\underline{3}} \cdot \underset{\text{Eng}}{\underline{1}} = 60$ ways

9\. a) $\underset{\text{Pres fem.}}{\underline{3}} \cdot \underset{\text{secr. male}}{\underline{2}} = 6$ ways b) $\underset{\text{pres male}}{\underline{2}} \cdot \underset{\text{secr. female}}{\underline{3}} = 6$ ways

Exercise 28: Counting Principles (continued)

9. c) $\rightarrow$ (a) or (b)
$= (a) + (b)$
$= 6 + 6 = 12$ ways

10.


$$\underset{A\text{ to }B}{5}\cdot\underset{B\text{ to }C}{4}\cdot\underset{C\text{ to }B}{3}\cdot\underset{B\text{ to }A}{4}$$
$= 240$ ways

11. $f(x) = x^{-3}$ or $y = x^{-3}$
Inverse: $x = y^{-3}$ $\therefore y^3 = \frac{1}{x}$ so $y = \frac{1}{x^{1/3}} = x^{(-1/3)}$ $\therefore$ c is the answer.

12. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$
L.H.S. $\sin^4 x - \cos^4 x$
$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$
$= \sin^2 x - \cos^2 x =$ R.H.S.

13. $\log_2 \sqrt[3]{4} = x$
$2^x = 4^{1/3}$
$2^x = (2^2)^{1/3}$
$x = 2/3$

14. $2(3^x) = 5^{x-1}$
$\log(2(3^x)) = \log 5^{x-1}$
$\log 2 + \log 3^x = (x-1)\log 5$
$\log 2 + x\log 3 = x\log 5 - \log 5$
$x(\log 3 - \log 5) = -\log 5 - \log 2$
$x(\log 3/5) = -(\log 5 + \log 2)$
$x(\log 3/5) = -\log 10$
$x = \frac{-\log 10}{\log 3/5}$
$x = \frac{-1}{-0.2218488} = +4.51$

15. LHS: $\log\left(\frac{2}{1.08}\right)$
$= \log 2 - \log 1.08$
$0.30103 - 0.033424$
$= 0.267606$

RHS: $\frac{\log 2}{\log 1.08}$
$= \frac{0.30103}{0.033424}$
$= 9.0064$ $\therefore$ LHS $\neq$ RHS

16. $4\cos^2\theta - 7\cos\theta - 2 = 0$
$(4\cos\theta + 1)(\cos\theta - 2) = 0$
$\cos\theta = -1/4$ $\cos\theta = 2$
no solution $\nearrow$
$\theta = 1.3181$ related angle
$\theta = 1.8235, 4.4597$

17. $S = S_0 e^{-0.04T}$
$25 = 50e^{-0.04T}$
$1/2 = e^{-0.04T}$
$\ln 1 - \ln 2 = -0.04T$
$\frac{-\ln 2}{-0.04} = T$
$T = 17.3$ years

18. $S = S_0 e^{-0.04t}$
$\frac{S}{S_0} = e^{-0.04t}$
$\ln\left(\frac{S}{S_0}\right) = -0.04t$
$\frac{\ln\left(\frac{S}{S_0}\right)}{-0.04} = t$
$-25\ln\left(\frac{S}{S_0}\right) = t$

Exercise 28: Counting Principles (continued)

19\. $P = P_0 e^{-KH}$

$89 = 101\cdot 3e^{-K(1)}$

$0\cdot 8785785 = e^{-K}$

$\ln 0\cdot 87858 = -K$

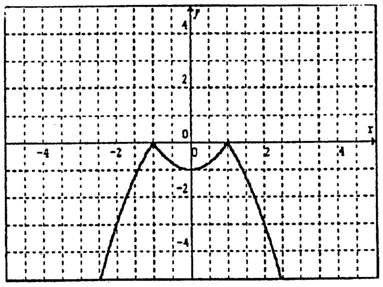
$0\cdot 12945 = K$

b) $P = 101.3e^{-0\cdot 12945(2)}$

$= 101.3e^{-0\cdot 2589}$

$P = 78.19$ KPa

20\.



Exercise 29: Permutation with Repetitions and Restrictions

1. a) $\frac{8!}{\underset{A}{\underset{\uparrow}{3!}}\,\underset{R}{\underset{\uparrow}{2!}}} = 3360$ b) $\frac{11!}{\underset{O}{\underset{\uparrow}{2!}}\,\underset{K}{\underset{\uparrow}{2!}}\,\underset{E}{\underset{\uparrow}{3!}}} = 1663200$

2. $\frac{11!}{\underset{\text{red}}{\underset{\uparrow}{3!}}\,\underset{\text{blue}}{\underset{\uparrow}{2!}}\,\underset{\text{green}}{\underset{\uparrow}{2!}}\,\underset{\text{yellow}}{\underset{\uparrow}{4!}}} = 69\,300$ 3. $\frac{\underset{H}{\underline{1}}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}}{\underset{\text{for T's}}{\underset{\uparrow}{2!}}\,\underset{\text{for E's}}{\underset{\uparrow}{2!}}} = 6$ ways

4. a) $\underset{R}{\underline{1}}\cdot\underline{5}\cdot\underline{4}\cdot\underline{3} = 60$ ways   b) $\underline{4}\cdot\underset{V}{\underline{2}}\cdot\underset{V}{\underline{1}}\cdot\underline{3} = 24$ ways

c) $\underset{C}{\underline{4}}\cdot\underset{C}{\underline{3}}\cdot\underset{C}{\underline{2}}\cdot\underset{C}{\underline{1}} = 24$ ways   d) $\left.\begin{array}{c}\underset{V}{\underline{2}}\cdot\underset{C}{\underline{4}}\cdot\underset{V}{\underline{1}}\cdot\underset{C}{\underline{3}} = 24\\ \text{or}\\ \underset{C}{\underline{4}}\cdot\underset{V}{\underline{2}}\cdot\underset{C}{\underline{3}}\cdot\underset{V}{\underline{1}} = 24\end{array}\right\}$ 48 ways

5. $\underset{\text{letter}}{\underline{26}}\cdot\underset{\text{letter}}{\underline{26}}\cdot\underset{\text{letter}}{\underline{26}}\cdot\underset{\text{digit}}{\underline{10}}\cdot\underset{\text{digit}}{\underline{10}}\cdot\underset{\text{digit}}{\underline{10}} = 26^3\cdot 10^3$ plates
(17576000)

6. a) $5\cdot5\cdot5 = 125$ b) The # must start with 1 or 3 or 5 and end with 5. This gives $3\cdot5\cdot1 = 15$ arrangements.

7. Amanda and Edna can be tied together in $2! = 2$ ways
The 4 objects can be arranged in $4! = 24$ ways
This leads to $2\cdot24 = 48$ permutations.

8. With no restrictions : $5! = 120$
Sitting together : 48
Not sitting together: $120 - 48 = 72$ ways

9. a) Each couple can be tied together in 2 ways.
The 4 couples can be seated in 4! ways.
This leads to: $2^4\cdot4! = 384$ ways.

Exercise 29: Permutation with Repetitions and Restrictions (continued)

9. b) The left hand seat can be occupied by either a man or woman.
The women can be seated in 4! ways.
The men can be seated in 4 ways.
This leads to $2 \cdot 4! \cdot 4! = 1152$ arrangements.

10. 4 E's and 3N's can be arranged in
$\frac{7!}{4!\,3!} = 35$ ways. Thus, there are 34 other ways from A to B.

11. $\log_{\frac{1}{49}} 7 = \frac{-1}{2}$

12. $\sec^4 x - \tan^4 x = 1 + 2\tan^2 x$
L.H.S. $\sec^4 x - \tan^4 x$
$= (\sec^2 x + \tan^2 x)(\sec^2 x - \tan^2 x)$
$= (1 + \tan^2 x + \tan^2 x)(1 + \tan^2 x - \tan^2 x)$
$= (1 + 2\tan^2 x)(1)$
$= 1 + 2\tan^2 x$ = R.H.S.

13. $\log_{100} 10 = x$
$100^x = 10$
$(10^2)^x = 10^1$
$2x = 1$
$x = \frac{1}{2}$

14. $\log_5 \left(\frac{mn^2}{p^3}\right)^3 = 3\log_5 \left(\frac{mn^2}{p^3}\right) = 3(\log_5 mn^2 - \log_5 p^3)$
$= 3(\log_5 m + 2\log_5 n - 3\log_5 P)$
or $3\log_5 m + 6\log_5 n - 9\log_5 P$

15. ① $x - y + 2 = 0 \rightarrow x = y - 2$
② $x^2 + y^2 - 4 = 0$
subst. into ② $(y-2)^2 + y^2 - 4 = 0$
$y^2 - 4y + 4 + y^2 - 4 = 0$
$2y^2 - 4y = 0$
$2y(y-2) = 0$
$y = 0$ or $y = 2$

① $x = y - 2$
$x = 0 - 2$ or $x = 2 - 2$
$x = -2$ $\quad$ $x = 0$
$\{(-2,0), (0,2)\}$

16. $\log_3 x = 2^2 = 4$
$x = 3^4 = 81$

Exercise 29: Permutation with Repetitions and Restrictions (continued)

17. $3x \log 5 = \log 63$

$3x = \dfrac{\log 63}{\log 5}$

$3x = 2.574274$

$x = 0.858091$

$x = 0.9$

or

$3x \log 5 = \log 63$

$x = \dfrac{\log 63}{3 \log 5}$

$x = 0.858091448$

$x = 0.9$

18. $\tan^2\theta - \tan\theta - 4 = 0 \rightarrow$ not factorable

$\tan\theta = \dfrac{-b \pm \sqrt{\Delta}}{2a}$, $a = 1$, $b = -1$, $c = -4$

$\Delta = b^2 - 4ac$

$= (-1)^2 - 4(1)(-4)$

$= 17$

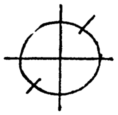
$\tan\theta = \dfrac{1 \pm \sqrt{17}}{2}$

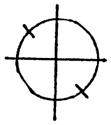
$\tan\theta = 2.5616$ or $\tan\theta = -1.5616$

Quads I and III Quads II + IV

$\theta = (\tan^{-1})2.5616$ $\theta = (\tan^{-1})-1.5616$

$\theta = 1.1988$ or 4.3403 $\theta = 5.2819$ or 2.1404





19. $(x-1)(x+1)(x-\frac{1}{2})(x-2) = (x^2-1)(x-\frac{1}{2})(x-2)$

$= (x^3 - 2x^2 - x + 2)(x - \frac{1}{2})$

$= x^4 - 2x^3 - x^2 + 2x - \frac{1}{2}x^3 \cancel{+x^2} + \frac{1}{2}x - 1$

$= x^4 - \frac{5}{2}x^3 + \frac{5}{2}x - 1$

$= 2x^4 - 5x^3 + 5x - 2$

20. $A = 800(3)^t$

ⓐ $A = 800(3)^{3.12}$

$A = 24643.8 = 24644$ bacteria

ⓑ $100000 = 800^{3t}$

$125 = 3^t$

$\log 125 = t \log 3$

$t = \dfrac{\log 125}{\log 3}$

$t = 4.39$ hours

Exercise 30: Permutations

1. a) $\frac{5!}{3!} = \frac{5\cdot4\cdot3!}{3!} = (5)(4) = 20$ b) $\frac{7!}{4!} = \frac{7\cdot6\cdot5\cdot4}{4!} = (7)(6)(5) = 210$

2. a) $\frac{n!}{(n-2)} = \frac{n(n-1)(n-2)!}{(n-2)!}$ $n(n-1)$

b)

$$\frac{\frac{n!}{(n-3)!}}{\frac{n!}{(n-2)!}} = \frac{\frac{n(n-1)(n-2)(n-3)!}{(n-3)!}}{\frac{n(n-1)(n-2)!}{(n-2)!}}$$

$$= \frac{n(n-1)(n-2)}{n(n-1)}$$

$$= n-2$$

3. $2n + \frac{n!}{(n-2)!} = 56$

$2n + n(n-1) = 56$

$n^2 + n - 56 = 0$

$(n+8)(n-7) = 0$

$n = 7$ $(n \neq -8$ since $n \geq 2)$

4. a) $6\cdot5\cdot4\cdot3\cdot2\cdot1 = 6! = 720$ ways

b) $\underset{B}{3}\cdot\underset{S}{3}\cdot\underset{B}{2}\cdot\underset{S}{2}\cdot\underset{B}{1}\cdot\underset{S}{1} = 36$

or

$\underset{S}{3}\cdot\underset{B}{3}\cdot\underset{S}{2}\cdot\underset{B}{2}\cdot\underset{S}{1}\cdot\underset{B}{1} = 36$

= 72 ways

5. $_{10}P_5 = \frac{10!}{5!} = 30,240$ ways

or

$\underset{\text{1st}}{10}\cdot\underset{\text{2nd}}{9}\cdot\underset{\text{3rd}}{8}\cdot\underset{\text{4th}}{7}\cdot\underset{\text{5th}}{6} = 30,240$ ways
student

6. Enter 4 ways. First person leaves in 9 ways. Second leaves in 8 ways.
$\therefore$ $4 \times 9 \times 8 = 288$ ways

7. $\underset{\text{ch.}}{8} \times \underset{\text{sec}}{7} \times \underset{\text{trea}}{6} = 336$ ways

8. a) $\underset{\text{odd}}{3}\cdot 2\cdot\underset{\text{odd}}{2}\cdot 1\cdot\underset{\text{odd}}{1} = 12$ ways b) $\underset{\text{odd}}{1}\cdot 2\cdot\underset{\text{odd}}{1}\cdot 1\cdot\underset{\text{odd}}{1} = 2$ ways

9. a) $8\cdot7\cdot6\cdot5 = 1680$ "words"

b) $7\cdot6\cdot5\cdot\underset{y}{1} = 210$ "words"

c) $\underset{\text{Not } R}{6}\cdot6\cdot5\cdot\underset{y}{1} = 180$ "words"

Exercise 30: Permutations (continued)

10. $_8P_3$ is the number of arrangements made using any 3 out of 8 available objects. $_3P_8$ does not make sense since it is impossible to arrange 8 objects if only 3 are available.

11. Since $-1 \le \cos x \le 1$, we conclude that $0 \le \cos^2 x \le 1$; thus the range of $f(x) = \cos^2 x$ is $[0, 1]$.

12. $5^{-1} = \frac{1}{5}$

13. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$

$$\begin{aligned}
L.H.S &= \frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{1} \\
&= \frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta} \\
&= \frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta} \\
&= \tan^2\theta \sin^2\theta \\
&= R.H.S
\end{aligned}$$

14. $\log_b x = n^2$ and $\log_x b = \frac{4}{n}$

$b^{n^2} = x$ $\quad x^{4/n} = b$

$(b^{n^2})^{4/n} = b$

$b^{4n} = b^1$

$4n = 1$

$n = 1/4$

15. $f(x) = e^{-x}$ is the graph of $y = e^x$ flipped in the y-axis

$g(x) = -e^x$ is the graph of $y = e^x$ flipped in the y-axis

$\therefore y = e^x$: Range$(0, \infty)$

y-int 1

$f(x) = e^{-x}$: Range $(0, \infty)$

y-int 1

$g(x) = -e^x$: Range $(-\infty, 0)$

y-int -1

16. $e^{2x-5} = 25$

$\ln e^{2x-5} = \ln 25$

$(2x-5)\ln e = \ln 25$

$2x - 5 = 3.22$

$2x = 8.22$

$x = 4.11$

or $2x - 5 = \ln 25$

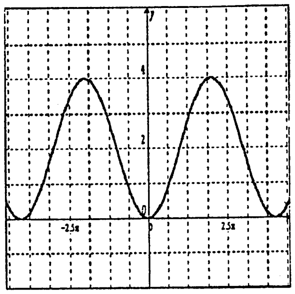
$2x = 5 + \ln 25$

$x = \frac{5 + \ln 25}{2}$

$x \approx 4.11$

Exercise 30: Permutations (continued)

17.

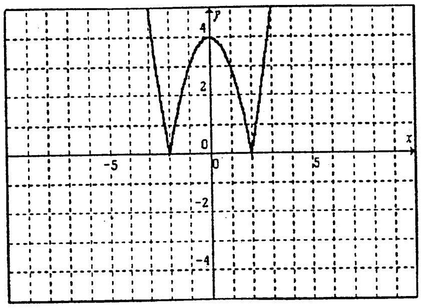


18. If $\sin x = \frac{1}{2}$, $0 < x < \pi$, then $x = \frac{5\pi}{6}$ or $\frac{\pi}{6}$

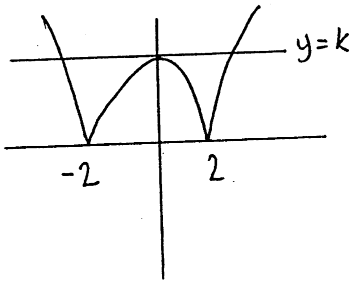
$AB = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6}$

$AB = \frac{2\pi}{3}$

19.



20. $K = 4$



$\therefore y = 4$

(The line must pass through the vertex of the parabola to intersect in exactly three places.)

Exercise 31: Circular Permutations

1\. a) Eight people in a circle $=(8-1)! = 7! = 5040$ ways.

b) Tie Bob + Raj. together. There are now 7 "groups." to arrange in a circle. This can be done in $(7-1)! = 6! = 720$ ways. Untie Bob & Raj. They can sit in 2 ways (Bob left or Raj left). $\therefore 720 \times 2 = 1440$ ways.

2\. Place the 5 men in a circle first. This can be done in $(5-1)! = 4! = 24$ ways. The 5 women fill the spaces between them in $5!$ ways $= 120$. $\therefore 24 \times 120 = 2880$

3\. row $= 12! = 479\ 001\ 600$

circle $= 11! = 39\ 916\ 800$

$\therefore$ The row has more seating arrangements.

4\. First, place the beads in a circle in $(4-1)! = 3!$ ways $= 6$ ways. Then, divide by 2 since the bracelet placed upside down appears as a different permutation. $\therefore$ 3 bracelets.

5\. Put the 3 friends together in $3! = 6$ ways and sit them down. Brad has only 5 choices for a chair. The remaining 5 people choose chairs from the 6 available.

$\therefore 6 \times 5 \times 6 \times 5 \times 4 \times 3 \times 2 = 21600$ ways. (Note, one chair remains empty.)

6\. Case 1: 5 first $\underset{5}{\underline{1}}\ \underset{6,8 \text{ or } 9}{\underline{3}}\ \underline{5}\ \underline{4} = 60$

Total $360 + 60 = 420$

Case 2: 6, 8 or 9 first $\underset{6,8 \text{ or } 9}{\underline{3}}\ \underline{6}\ \underline{5}\ \underline{4} = 360$

7\. a) $\underset{\text{Not } 0}{\underline{6}} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} = 720$

b) Case 1: ends in 5: $\underset{\text{not } 0}{\underline{5}}\ \underline{5}\ \underline{4}\ \underset{5}{\underline{1}} = 100$

Total 220

Case 2: ends in 0: $\underline{6}\ \underline{5}\ \underline{4}\ \underset{0}{\underline{1}} = 120$

Exercise 31: Circular Permutations (continued)

7.c) Case 1: ends in 0: $\underline{6}\cdot\underline{5}\cdot\underline{4}\cdot\frac{1}{0} = 120$ ∴ Total = 420

Case 2: ends in 2, 6, or 8: $\underset{\text{not } 0}{5}\cdot 5\cdot 4\cdot \underset{\text{2, 6 or 8}}{3} = 300$

or Total no. of odd no. is $\underset{\text{not } 0}{5}\ \underline{5}\ \underline{4}\ \underset{\text{1, 3 or 5}}{3} = 300$ ∴ $\underset{\text{(from A)}}{720} - 300 = \underset{\text{No. of evens}}{420}$

8. 3 digit $\underset{<7}{\underline{6}}\cdot\underline{9}\cdot\underline{8} = 432$
+
or 2 digit $\underline{9}\cdot\underline{9} = 81$
+
or 1 digit $10 = \underline{10}$
523

∴ 523 numbers less than 700 have no repetition of digits.

9. a) $\frac{7!}{3!\,2!} = 420$

b) Case 1: first is 3. Second must be 4 or 5 ∴ $2\times\frac{5!}{3!} = 40$

Case 2: first is 4 or 5: $2\times\frac{6!}{3!\,2!} = 120$ ∴ total = 160

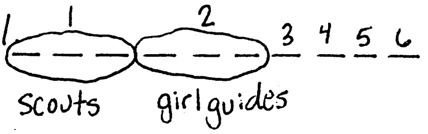
c) Case 1: first is 3, second is 4, last is 5: $\frac{4!}{3!} = 4$ ways

Case 2: first is 4, last is 5: $\frac{5!}{3!\,2!} = 10$ ways ∴ 14 numbers div. by 5

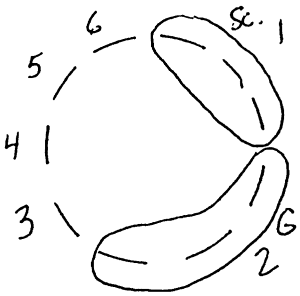
10. End men chosen in 5 × 4 = 20 ways. The remaining 3 men + 3 women sit in 6! ways. ∴ 20 × 6! = 14 400 ways.

Exercise 31: Circular Permutations (continued)

11.



$6! \cdot 3! \cdot 3! = 25920$ ways



$= 5! \cdot 3! \cdot 3!$
$= 4320$ ways

Tie Scouts together; tie the Guides together. Permute 6 "groups" in 6! ways. Untie the groups Each can arrange themselves in 3! ways $\therefore$ $6!\ 3!\ 3! = 25920$ ways

Again tie them in groups. Place the Scouts in chairs. There are 5 more "groups" to place in 5! ways. Again, untie the group and permute each in 3! ways.
$\therefore 5! \cdot 3! \cdot 3! = 4320$ ways.

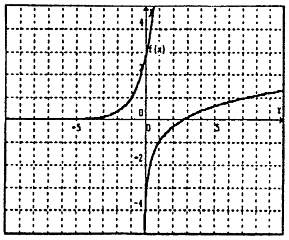
a) The class in period one has the greater number of arrangements.
b) There are $25920 - 4320 = 21,600$ more ways to sit in a row.

12. $(\sin^2 x + \cos^2 x)^6 = 1$
$L.H.S = (\sin^2 x + \cos^2 x)^6$
$= (1)^6$
$= 1$
$= R.H.S.$

13. $\log_6 3^2 + \log_6 16^{1/2}$
$\log_6 9 + \log_6 4$
$\log_6 (9 \times 4) = \log_6 36$
$= \log_6 (6)^2 = 2 \log_6 6 = 2$

14. $e^{x^2} = e^{x + 3/4}$
$x^2 - x - 3/4 = 0$
$4x^2 - 4x - 3 = 0$
$(2x-3)(2x+1) = 0$
$x = 3/2$ and $x = -1/2$

15.



$y = e^{x+1}$
$x = e^{y+1}$
$\ln x = y + 1$
$-1 + \ln x = y = f^{-1}(x)$

Exercise 31: Circular Permutations (continued)

$$16.\ \frac{(n+2)(n+1)(n)(n-1)\cancel{(n-2)}!}{8!\ \cancel{(n-2)}} = \frac{57}{16} \Rightarrow \qquad (n+2)(n+1)(n)(n-1) = \frac{3\cdot 19\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 8}$$

We need 4 consecutive numbers (from the left side) whose factors are those on the right. Try to create numbers around the factor 19. Notice that $3 \times 6 = 18$, $4 \times 5 = 20$, $7 \times 3 = 21$.

$\therefore (n+2)(n+1)(n)(n-1) = 21 \cdot 20 \cdot 19 \cdot 18 \qquad \therefore n = 19$

$$17.\ \frac{(n+1)(n)(n-1)!}{(n-1)!} = 30 \Rightarrow n^2+n = 30 \Rightarrow n^2+n-30 = 0$$

$$\Rightarrow (n+6)(n-5) = 0 \Rightarrow \cancel{n=-6} \text{ or } n=5 \therefore n=5$$

since $n \geq 1$

$18.\ \sin x = -1 \ \therefore x = \frac{3\pi}{2} + 2k\pi$, k is an integer.

19. a) $\sin(x+20) = \sin x \cos \pi + \cos x \sin 2\pi$

$= \sin x(1) + \cos x(0)$

$= \sin x$

b) $\sin(-x) = \sin(0-x) = \sin 0 \cos x - \cos 0 \sin x$

$= 0(\cos x) - (1)\sin x$

$= -\sin x$

c) $\cos(\frac{\pi}{2}-x) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= (0)\cos x + (1)\sin x$

$= \sin x$

d) $\sin(x-6\pi) = \sin x \cos 6\pi + \cos x \sin 6\pi$

$= \sin x(1) + \cos x(0)$

$= \sin x$

$\therefore$ B is the only expression not equal to $\sin x$.

20. a) $\cos(x+2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi$

$= \cos x(1) - \sin x(0)$

$= \cos x$

b) $\cos(-x) = \cos(0-x) = \cos 0 \cos x + \sin 0 \sin x$

$= (1)\cos x + (0)\sin x$

$= \cos x$

c) $\sin(\pi/2 - x) = \sin\pi/2 \cos x - \cos\pi/2 \sin x$

$= (1)\cos x - (0)\sin x$

$= \cos x$

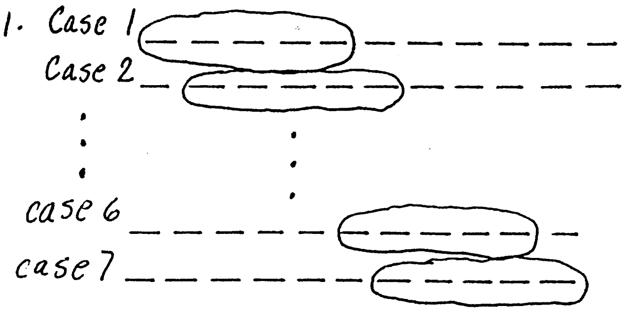
d) $\cos(x+\pi) = \cos x \cos\pi - \sin x \sin\pi$

$= \cos x(-1) - \sin x(0)$

$= -\cos x$

$\therefore$ D is the only expression not equal to $\cos x$.

Exercise 32: Permutations with Case Restrictions

1\.


Five consecutive chairs can be placed in 7 ways. The 5 people can be seated in 5! ways. $\therefore 7 \times 5! = 7 \times 120 =$ 840 ways.

2\.

$5!$	$3!$	$3!$	$1!$
Dickens	Shake	Dan Steele	Marg.L

$\therefore$ Total arrangements $=$

$\underset{\text{each group of books}}{4!} \cdot 5! \cdot 3! \cdot 3! \cdot 1!$

$= 103,680$

3\. a) Tie the 2 together. Place 7 "people" in a circle in $(7-1)! = 6! = 720$ ways. Untie the 2 + permute them in 2 ways $\therefore 2 \times 6! = 1440$ ways.

b) Tie Brent and Nicky together and George and Monica together. Permute the 6 "people" in 5! ways. Untie the couples. This gives $2 \times 2 \times 5! = 480$ permutations. Subtract from the 1440 ways in part A. $1440 - 480 = 960$ ways.

4\. a) $\underset{\text{boys}}{\underline{4}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underset{\text{boys}}{\underline{3}} = \underset{\text{ways}}{288}$

b) Boys can be arranged 4! ways. The 2 girls and the group of boys can be arranged in $3! = 6$ ways. This leads to $(4!)(6) = 144$ arrangements.

5\. a) $\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} = 720$ ways

b) $\underline{5} \cdot \underset{I}{\underline{1}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} = 120$ ways

c) use complements

$\underset{\text{(Total \# of ways)}}{720} - \underset{E}{1} \,.\, \underset{\text{(when it starts with E)}}{5 \cdot 4 \cdot 3 \cdot 2} = 600$ ways

6\. a) $\dfrac{7!}{3!} = 840$ ways

b) $\dfrac{\overset{R}{\underline{3}}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}}{3!} = 360$ ways

c) $\dfrac{\overset{R}{\underline{3}}\ \overset{\text{Not } R}{\underline{4}}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}}{3!} = 240$ ways

Exercise 32: Permutations with Case Restrictions (continued)

7. No. of ways the unruly boys can be seated in the middle 6 seats $= \underset{\text{boy 1}}{6} \cdot \underset{\text{boy 2}}{5} \cdot \underset{\text{others}}{6!} = 21\,600$ - Total - not on end.

No. of ways the unruly boys are together in the middle seats is $\underset{\text{ways of choosing 2 seats}}{5} \cdot \underset{\text{left or right}}{2} \cdot \underset{\text{others}}{6!} = 7200.$

$21600 - 7200 \therefore 14\,400$ ways

8. Case 1: 5, 3 First $\frac{1}{5}\ \frac{1}{3}\ \frac{2}{7\text{or }8}\ \frac{3}{\ } = 6$

Case 2:

5 First, 7 or 8 second $\frac{1}{5}\ \frac{2}{7\text{or }8}\ \frac{4}{\ }\ \frac{3}{\ } = 24 \therefore$ Total is 150

Case 3: 5 Not first $\frac{2}{7\text{or }8}\ \frac{5}{\ }\ \frac{4}{\ }\ \frac{3}{\ } = 120$

9. $\underbrace{\frac{5}{1} \cdot \frac{4}{2} \cdot \frac{3}{3} \cdot \frac{2}{4}}_{\text{boys}} \cdot \underbrace{\frac{7}{1} \cdot \frac{6}{2} \cdot \frac{5}{3} \cdot \frac{4}{4} \cdot \frac{3}{5}}_{\text{girls}} = 302\,400$ ways

10. a) 12 people in a circle $11! = 39\,916\,800$ ways

b) $6!\,6! = 518\,400$. Pick the left-most girl in 6 ways, the next girl in 5 ways, etc. This gives $6!$ ways for the girls and then $6!$ ways for the boys.

c) Girls in a circle in $5!$ ways. Boys between them in $6!$ ways. $\therefore 5!\,6! = 86\,400$ ways

11. Using the identity $\sin(A-B) = \sin A \cos B - \cos A \sin B$ we get $\sin, \left(\frac{\pi}{16} - \frac{3\pi}{16}\right) = \sin\left(\frac{-2\pi}{16}\right) = \sin\left(\frac{-\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

Exercise 32: Permutations with Case Restrictions (continued)

12. $(2^{2(x)} = 2^6$
$2x = 6$
$x = 3$

13. $\pi < 4 < \frac{3}{2}\pi$ so $P(4)$ is in Quadrant $\underline{III}$
or $3.14 < 4 < 4.71$

14. a) $9^x = \frac{1}{27}$
$(3^2)^x = (3)^{-3}$
$2x = -3$
$x = -3/2$

b) $(3^3)^x = 3^1$
$3x = 1$
$x = 1/3$

15. a) $f(x) = (1/2)^{x-4}$
$= (2^{-1})^{x-4} = 2^{-(x-4)}$
∴ a and C are equivalent

16. $\log_b 2t + \log_b 3t = \log_b 24$
$\log_b 6t^2 = \log_b 24$
$6t^2 = 24$ $t^2 = 4$ $t = -2$ $t = -2$
reject
∴ $t = 2$

17. $2\sin^2\theta + \cos\theta = 1$
$2(1 - \cos^2\theta) + \cos\theta = 1$
$2 - 2\cos^2\theta + \cos\theta = 1$
$2\cos^2\theta - \cos\theta - 1 = 0$
$(2\cos\theta + 1)(\cos\theta - 1) = 0$
$\cos\theta = 1/2$ $\cos\theta = 1$
$\theta = \pi/3$ related angle $\theta = 0, 2\pi$
$\theta = 2\pi/3, 4\pi/3$
$\theta = 0, 2\pi/3, 4\pi/3, 2\pi$

18. $T = 20 + 80e^{-.03t}$
$40 = 20 + 80e^{-.03t}$
$20 = 80e^{-.03t}$
$1/4 = e^{-.03t}$
$\ln 1 - \ln 4 = -0.03t$
$\frac{-\ln 4}{-0.03} = t$
$t = 46.21$ minutes

19. Two possible equations: standard form $(y = a\sin b(x-c)+d)$
sine EQ. $y = -2\sin 2(x - \frac{\pi}{12}) + 5$

$a = 2, b = 2, c = \frac{5\pi}{12}$ (mid pt. of $\pi/6$ + $2\pi/3$) $d = 5$
cosine EQ. $y = 2\cos 2(x + \pi/6) + 5$
$a = 2$, Period π, $b = 2$, $c = \pi/6$, $d = 5$

20. a) $i = 5\cos 120\pi t$ where $a = 5$, period $= 1/60$
$b = 120\pi$
no horizontal

b) $i = 5\cos 120\pi(0.01)$
$i = -4.045$ amperes

Exercise 33: Combinations

1. a) ${}_4C_2 = \frac{4!}{2!2!} = 6$

b) ${}_6C_3 = \frac{6!}{3!3!} = 20$

c) ${}_7C_5 = \frac{7!}{5!2!} = 21$

2. a) ${}_8C_2 = \frac{(8)(7)}{2} = 28$

${}_8C_6 = \frac{(8)(7)(6)(5)(4)(3)}{6!} = 28$

b) ${}_5C_2 = \frac{(5)(4)}{2} = 10$

${}_5C_3 = \frac{(5)(4)(3)}{3} = 10$

c) In each pair, the answers are the same

d) If we wish to take objects from N, this can be achieved by leaving behind n-r objects. Thus $nCr = nCn-r$.

3. a) $n = 2+8 = 10$

b) $(x+1)+(x+3) = 12$

$2x+4 = 12$

$2x = 8$

$x = 4$

4. ${}_{49}C_6 = \frac{49!}{6!\,43!} = 13\ 983\ 816$ ways

A baseball nine consists of 1 catcher, 1 pitcher, 4 infielders and 3 out fielders.

5. $\underset{\text{catcher}}{{}_3C_1} \cdot \underset{\text{pitcher}}{{}_5C_1} \cdot \underset{\text{infielders}}{{}_7C_4} \cdot \underset{\text{out fielders}}{{}_7C_3} = 18\,375$ baseball nines

6. a) $\underset{\text{all cards}}{{}_{52}C_5} = 2,598\ 960$ hands

b) $\underset{\text{red cards}}{{}_{26}C_5} = 65,780$ hands

c) $\underset{\text{hearts}}{{}_{13}C_2} \cdot \underset{\text{clubs}}{{}_{13}C_2} \cdot \underset{\text{spades/diamonds}}{{}_{26}C_1} = 158184$

7. ${}_5C_3 + {}_5C_4 + {}_5C_5 = 10 + 5 + 1 = 16$ sums

8. a) ${}_4C_2 \cdot {}_6C_1 = 36$ ways

b) ${}_4C_1 \cdot {}_6C_2 = 60$

or ${}_4C_2 \cdot {}_6C_1 = +36$

or ${}_4C_3 \cdot \quad = +4$

100 ways

c) ${}_4C_3$ (all women) $= 4$

or +

${}_6C_3$ (all men) 20

24 ways

Exercise 33: Combinations (continued)

9\. a) $\underset{\text{black balls}}{{}_4C_1} \cdot \underset{\text{white balls}}{{}_7C_2} = 84$ ways b) $\underset{\text{all black}}{{}_4C_3} + \underset{\text{all white}}{{}_7C_3} = 39$ ways

c) $\underset{\text{1 Black ball}}{{}_4C_1 \cdot {}_7C_2} = 84$

+

or $\underset{\text{2 black balls}}{{}_4C_2 \cdot {}_7C_1} = 42$

+

or $\underset{\text{all black}}{{}_4C_3} = 4$

$\overline{130 \text{ ways}}$

10\. a) ${}_9C_6 = \frac{9!}{6!\,3!} = 84$ ways

b) case1 Neither comes: ${}_7C_6 = 7$ways

case2 1comes $2 \times {}_7C_5 = 2 \times 21 = 42$

∴ Total = 49

11\. $4\left(\frac{n!}{(n-3)!}\right) = 5\left(\frac{(n-1)!}{(n-4)!}\right)$

$\frac{4n!}{(n-3)!} = \frac{5(n-1)!}{(n-4)!}$

$\frac{4n(n-1)!}{(n-3)(n-4)!} = \frac{5(n-1)!}{(n-4)!}$

$\frac{4n}{n-3} = 5$

$4n = 5n - 15$

$n = 15$

12\. There are ten ways to choose the child in the center. The other 9 children can be arranged in a circle in 8! ways.

$\therefore (10)(8!) = 403\,200$

13\. $\frac{1+\tan^2\theta}{1+\cot^2\theta} = \sec^2\theta - 1$

$L.H.S = \frac{1+\tan^2\theta}{1+\cot^2\theta}$

$= \frac{\sec^2\theta}{\csc^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

$= \sec^2\theta - 1$

$= R.H.S.$

14\. $\frac{12!}{5!\,2!} = 1995\,840$

$\uparrow\ \uparrow$

B A

15\. $\log_2(x-1) + \log_2(x+2) = 2$

$\log_2 (x-1)(x+2) = 2$

$2^2 = (x-1)(x+2)$

$4 = x^2 + x - 2$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = -3 \quad x = 2$ ✓

check extraneous root

Exercise 33: Combinations (continued)

16. $\log_5 5^2 + \log_2 512^{1/3}$

$\log_5 5^2 + \log_2 8$

$\log_5 5^2 + \log_2 2^3 = 2\log_5 5 + 3\log_2{}^2$

$= 2 + 3 = 5$

17. a) $\dfrac{14!}{\underset{\overset{\uparrow}{C}}{2!}\ \underset{\overset{\uparrow}{I}}{4!}\ \underset{\overset{\uparrow}{N}}{3!}\ \underset{\overset{\uparrow}{O}}{2!}} = 151\,351\,200$

b) T First $\dfrac{13!}{2!\,4!\,3!\,2!} = 10\,810\,800$

18. Case 1: 5 First, 8 second $\frac{1}{5}\ \frac{1}{8}\ 4\ 3 = 12$

Case 2: 6 or 8 First $\frac{2}{6 \text{ or } 8}\ 5\ 4\ 3 = 120$ $\therefore$ Total $= 132$

19. $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-4/9} = \dfrac{\sqrt5}{3}$ or

$opp^2 + adj^2 = hyp^2$

$3^2 + adj^2 = 2^2$

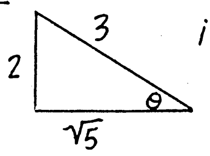
$9 - 4 = adj^2$

$5 = adj^2$

$\pm\sqrt5$ $\therefore \cos\theta = \dfrac{\sqrt5}{3}$ in QI

a) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{2}{3}\right)\left(\frac{\sqrt5}{3}\right) = \dfrac{4\sqrt5}{9}$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{5}{9} - \frac{4}{9} = \frac{1}{9}$



20. a) $\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= \dfrac{4\sqrt5}{9}\cdot\dfrac{\sqrt5}{3} + \dfrac{1}{9}\cdot\dfrac{2}{3} = \dfrac{22}{27}$

b) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= \dfrac{1}{9}\cdot\dfrac{\sqrt5}{3} - \dfrac{4\sqrt5}{9}\cdot\dfrac{2}{3}$

$= \dfrac{-7\sqrt5}{27}$

Exercise 34: Binomial Theorem

1. a) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

b) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

2. $(2x-y)^4 = (2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$

$= 16x^4 - 32x^3y + 24x^2y - 8xy^3 + y^4$

(Note that the signs alternate)

3. a) $(2x+1)^9 = (2x)^9 + {}_9C_8(2x)^8(1) + {}_9C_7(2x)^7(1)^2 = 512x^9 + 2304x^8 + 4608x^7$

b) $(2x^2)^{11} + {}_{11}C_{10}(2x^2)^{10}(-x) + {}_{11}C_9(2x^2)^9(-x)^2 = 2048x^{22} - 11264x^{21} + 28160x^{20}$

4. ${}_7C_3(3a^2)^4\left(\frac{-3}{a}\right)^3$

$= 35(81a^8)\left(-\frac{27}{a^3}\right)$

$= -76545a^5$

5. ${}_{10}C_4\, a^4b^6$

$\frac{10!}{6!4!}a^4b^6$

$210a^4b^6$

6. ${}_7C_3\left(\frac{y}{4}\right)^3\left(\frac{-2}{y}\right)^4 = (35)\left(\frac{y^3}{64}\right)\left(\frac{16}{y^4}\right) = \left(\frac{35}{4y}\right)$

7. Middle term is the 7th term $= {}_{12}C_6(2x)^6\left(-\frac{1}{2x}\right)^6 = 924$

8. 6th term is ${}_{11}C_5(2y)^6x^5 = 29\,568\,y^6x^5$

9. $(r+1)^{st}$ term is ${}_9C_r(3a)^{9-r}\left(\frac{-1}{6a^2}\right)^r$

10. 1st term has x^{14} 2nd term has $x^{13}x^4 = x^{17}$

3rd term has $x^{12} \cdot x^8 = x^{20}$

3rd term is ${}_{14}C_{12}(2x)^{12}(-x^4)^2$ $-372\,736\,x^{20}$

11. $({}_{15}C_7)({}_8C_4)({}_4C_4) = (6435)(70)(1) = 450\,450$ ways

left middle right

Exercise 34: Binomial Theorem (continued)

12. a) Choose the 6 beads ${}_{10}C_6 = 210$ ways.

b) Form a circular permutation in $5! = 120$ ways.

c) Divide by 2 since it is a bracelet $\frac{210 \times 120}{a} = 12\ 600$ ways.

13. a) $10 \cdot 8 = 80$ ways b) $({}_{10}C_2)\cdot({}_8C_2) = (45)(28) = 1260$ ways

c) $({}_{10}C_2)\cdot({}_8C_2)\cdot \underset{\uparrow}{2} = 2520$ ways

ways to make pairs

14. a) $\underset{\text{vowels}}{{}_5C_2} \cdot \underset{\text{consonents}}{{}_{21}C_3} = 13,300$ ways

b) $\underbrace{{}_5C_2 \cdot {}_{21}C_3}_{\text{choose}} \cdot \underset{\text{arrange}}{5!} = 1,596\ 000$ ways

c) $[({}_5C_2 \cdot {}_{21}C_3) + ({}_5C_1 \cdot {}_{21}C_4) + ({}_{21}C_5)] \cdot 5!$

choose 3cons. or 4 cons. or 5cons. arrange.

15. a) First, seat the 2 opposite 1 way, then put the 2 together. There are 2 possible chair choices 2 ways (R-L or L-R) to sit = 4 ways. The 2 apart? Impossible. There is no way these people can sit in 6 chairs.

b) Set the opposite in 1 way. This leaves two blocks of 3 chairs. Set the 2 together in 4x2 (choices of 2 chairs x order of couple) = 8 ways. The 2 apart have 8 ways to sit (one with the together group + the other in one of the 3 chairs opposite. or both in the block of 3 chairs). ∴ 4x8 = 32 ways.

Exercise 34: Binomial Theorem (continued)

16. $\frac{\tan x}{\sec x-1} = \frac{\sec x+1}{\tan x}$

L.H.S. $\frac{\tan x}{\sec x-1}$

$= \frac{\tan x}{\sec x-1} \cdot \frac{\sec x+1}{\sec x+1}$

$= \frac{\tan x(\sec x+1)}{\sec^2 x-1}$

$= \frac{\tan x(\sec x+1)}{\tan^2 x}$

$= \frac{\sec x+1}{\tan x} = R.H.S.$

17. a) $y = 10\,000e^{.6x}$

$y = 10\,000e^{.6(7)}$

$= 666863.31$

b) $y = 10000e^{.6x}$

$30000 = 10000e^{.6x}$

$3 = e^{.6x}$ $\ln 3 = .6x$

$x = \frac{\ln 3}{.6} = 1.83$ days

18. a) Two teams are needed for a game. ${}_{23}C_2 = 253$ games

b) To play each team twice as many games

$= 2 \times 253 = 506$ games.

19. $2\cos\theta = 3\tan\theta$

$2\cos\theta = 3\left(\frac{\sin\theta}{\cos\theta}\right)$

$2\cos\theta = \frac{3\sin\theta}{\cos\theta}$

$2\cos^2\theta = 3\sin\theta$

$2(1-\sin^2\theta) = 3\sin\theta$

$2-2\sin^2\theta = 3\sin\theta$

$2\sin^2\theta + 3\sin\theta - 2 = 0$

$(2\sin\theta-1)(\sin\theta+2) = 0$

$2\sin\theta = 1$ $\quad$ $\sin\theta = -2$

$\sin\theta = \frac{1}{2}$ $\quad$ no solution

$\theta = \pi/6$ related angle

$\theta = \pi/6, 5\pi/6$

20. a) $A = P(1+\frac{r}{n})^{nt}$

$A = 4000(1+\frac{.08}{1})^{1\times 10}$

$= 4000(1.08)^{10}$

$= \$8635.70$

b) $A = 4000(1+\frac{.08}{4})^{4\times 10}$

$A = \$8\,832.16$

c) $A = 4000(1+\frac{.08}{12})^{12\times 10}$

$A = \$8878.56$

d) $A = 4000(1+\frac{.08}{365})^{365\times 1}$

$A = \$8901.38$

e) $A = Pe^{rt}$

$A = 4000e^{(.08)(10)}$

$= 4000e^{.8}$

$= \$8901.16$

Exercise 35: Permutations, Combinations, and Binomial Theorem

1. ${}_8C_3 \cdot {}_{10}C_2 = 2520$
retrievers huskies +
or ${}_8C_4 \cdot {}_{10}C_1 = 700$
+
${}_8C_5 = 56$
∴ 3276 teams

2. a) $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$
or ${}_{12}P_9 = 79,833,600$ ways
b) $2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$
Pitcher 3 628 800 ways
c) $2 \cdot 11 \cdot 10\ 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 =$
Pitcher 13 305 600 ways

3 $\left(2x - \frac{1}{x}\right)^3 = (2x)^3 - 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^3$
$= 8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}$

4. ${}_{20}C_9\,(x)^{11}(-2y)^9$

5. 1st term has $(x^5)^{10} = x^{50}$ 2nd term has $(x^5)^9 \cdot \frac{1}{x^3} = x^{42}$
exp decreases by 8
∴ the term in x^2 is the 7th term
7th term is ${}_{10}C_6\left(\frac{x^5}{2}\right)^4 \left(\frac{-2}{x^3}\right)^6 = 210\left(\frac{x^{20}}{16}\right)\left(\frac{+64}{x^{18}}\right) = 840x^2$

6. 1st term has $(x^4)^{12} = x^{48}$
2nd term has $(x^4)^{11} \cdot \frac{1}{x^2} = x^{42}$ exp decreases by 6
9th term has no x
9th term $= {}_{12}C_4\ (2x^4)^4\left(-\frac{1}{2x^2}\right)^8 = 495(16x^{16})\left(\frac{1}{256x^{16}}\right) = \frac{495}{16}$

7. 1st term x^{11} 2nd term $x^{10} \cdot x^2 = x^{12}$ exp. increases by 1.
∴ term in x^{14} is the 4th term
4th term $= {}_{11}C_8\ (2x)^8(-x^2)^3 = (165)(256x^8)(x^6) = -42240x^{14}$

8. a) If the president is included, then 4 more members come from the 8. ${}_8C_4 = 70$
b) If the president is excluded, 5 must be chosen from $8 = {}_8C_5 = 56$.

9. a) If the top scorer is included, then 2 players from the other 8 are needed. ${}_8C_2 = 28$ lines.
b) Lines without Marie + Serge at all $= {}_7C_3 = 35$. Therefore, lines with them = total no. of possible lines $- 35 = {}_9C_3 - 35 = 84 - 35 = 49$ lines.

Exercise 35: Permutations, Combinations, and Binomial Theorem (continued)

9. c) 1.

10. Case 1. Exactly 2A's : choose 3 letters from the remaining 5 in $5C_3 = 10$ ways
Permute them all in $\frac{5!}{2!} = 60$ ways. $\therefore 10 \times 60 = 600$ ways.
Case 2: All Different : $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$. $\therefore$ Total $= 1320$ ways.

11. To form a rectangle we need 2 vertical lines and 2 horizontal lines. $\therefore$ $\underset{(V)}{{}_7C_2} \cdot \underset{(H)}{{}_4C_2} = 126$ rectangles.

12. Recall: ${}_nC_r = \frac{n!}{(n-r)!r!}$

$(n+2)C_4 = 6({}_nC_2)$

$\frac{(n+2)!}{(n+2-4)!4!} = 6\left(\frac{n!}{(n-2)!2!}\right)$

$\frac{(n+2)!}{\cancel{(n-2)!}} = \frac{4!6 \cdot n!}{\cancel{(n-2)!}2!}$

$(n+2)! = 72n!$

$\frac{(n+2)!}{n!} = 72$

$\frac{(n+2)(n+1)\cancel{(n!)}}{\cancel{n!}} = 72$

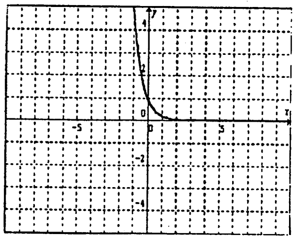
$(n+2)(n+1) = 72$

$n^2 + 3n - 70 = 0$

$(n+10)(n-7) = 0$

$\cancel{n = -10}$ $n = 7$

13.



Domain $\{x | x \in R\}$
Range: $\{y | y > 0\}$
Horizontal Asymptote : $y = 0$
No x-intercept
y-intercept is 1
The graph of $f(x)$ and $g(x)$ are the same.

14. $\log_{2a}(4a^2)^3 = x$
$(2a)^x = (4a^2)^3$
$(2a)^x = 64a^6$
$(2a)^x = (2a)^6$
$x = 6$

15. $\log_x \frac{(9.3 \times 8.6)}{19.1}$

Exercise 35: Permutations, Combinations, and Binomial Theorem (continued)

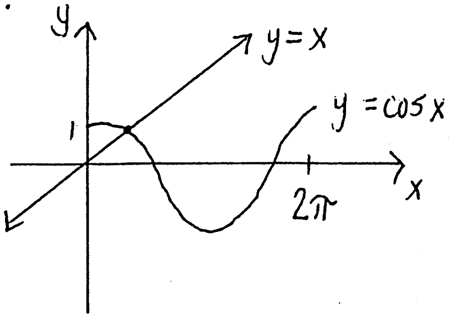
16. Seat the host and hostess and 4 men in $2 \times 4! = 48$ ways
Case 1: No wife sits on the same side of the table as her husband $= 2! \cdot 2! = 4$ ways.
Case 2: One wife sits on the same side as her husband. Since 2 wives must sit across from their husbands, there are only 2 wives who can sit on the same side. They can only do this in 1 way. Total way to seat wives is $4+1 = 5$.
$\therefore$ Total $= 5 \times 48 = 240$ ways.

17. $\dfrac{7!}{3!\,2!\,2!} = 210$

18. ${}_9C_2 = 36$

19. a) $1.\ {}_{11}C_4 = 330$
b) ${}_{11}C_5 = 462$

20.

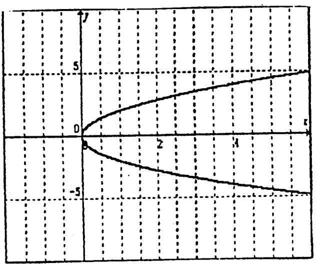


The graph shows that there is only one solution to $x = \cos x$. Using the "solver" feature on a calculator, we get $x = 0.739$

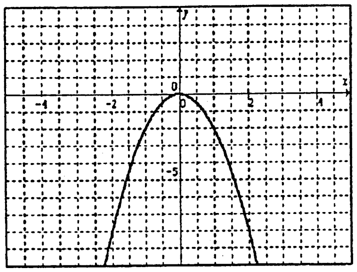
(a variety of other calculator methods are available)

Exercise 36: Parabola

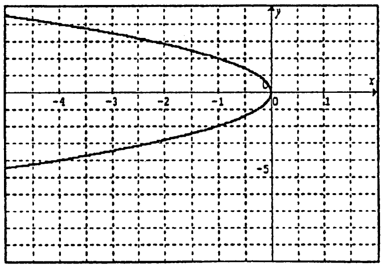
1.a)



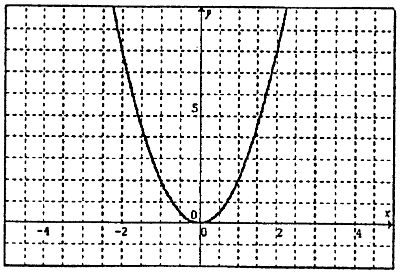
b)



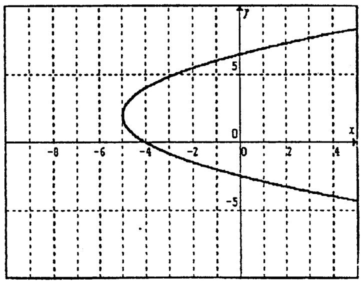
c)



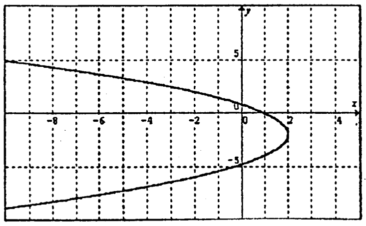
d)



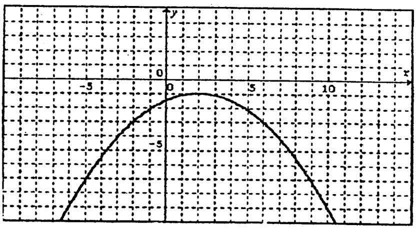
2 a)



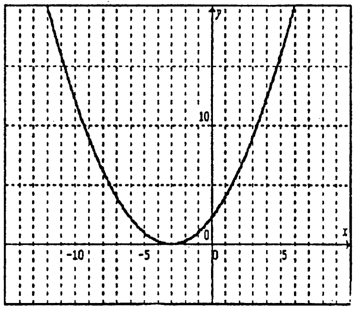
2.b)



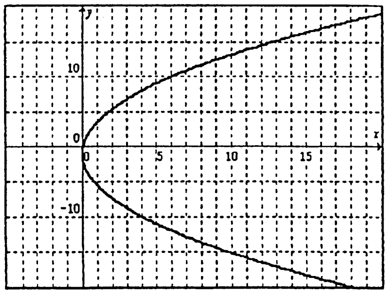
3.a)



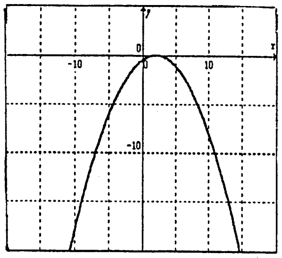
b)



4.



5.



Exercise 36: Parabola (continued)

6. $5 = a(2^2)$

$a = \frac{5}{4}$

7. $1^2 - 12 + 4(1) + k = 0$

$-7 + k = 0$

$k = 7$

$y^2 + 4y + 7 = x$

$y^2 + 4y + 4 = x - 3$

$(y + 2)^2 = (x - 3)$

vertex $(3, -2)$

8. $y = 2$

9. $x = -2$

10. $y + 3 = a(x-2)^2$

$-10 + 3 = a(9-2)^2$

$-7 = 49a$

$a = -\frac{1}{7}$

$\therefore y + 3 = -\frac{1}{7}(x-2)^2$

11. $6! \times 2 = 1440$

12. Case 1 $\quad {}_7C_4 \times {}_6C_1 = 210$ (4 retrievers)

Case 2 $\quad {}_7C_5 = 21$ (5 retrievers)

Total = 231 teams.

13. $\frac{{}_8C_5 \times 4!}{2} = 672$

14. $2^{-(x+1)} = 2^6$

$-x - 1 = 6$

$-x = 7 \quad x = -7$

15. $5^2 = 25$

16. $\log_3 x = 3 - \log_3 (x+6)$

$\log_3 x + \log_3 (x+6) = 3$

$\log_3 x(x+6) = 3$

$3^3 = x(x+6)$

$27 = x^2 + 6x$

$0 = x^2 + 6x - 27$

$0 = (x+9)(x-3)$

$x = -9_{x} \qquad x = 3$

Check: extraneous root

17. First term is x^{12};

second term is $(x^{11})\left(\frac{1}{2x}\right)^1 = x^{10}$.

The pattern goes down by 2.

$\therefore$ you need 7 terms.

${}_{12}C_6 (x)^6 \left(\frac{1}{2x}\right)^6$

$= 924 \left(\frac{x^6}{64x^6}\right)$

$= \frac{231}{16}$

Exercise 36: Parabola (continued)

18. L.H.S. =

$$\frac{\cos\theta}{\frac{1}{\sin\theta}} - \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= \cos\theta \sin\theta - \frac{\sin\theta\cos\theta}{\sin\theta}$$

$$= \cos\theta \sin\theta - \cos\theta$$

$$= \cos\theta(\sin\theta - 1)$$

$$= \frac{\sin\theta - 1}{\sec\theta}$$

$$= R.H.S.$$

19. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$2 = 1\left(1 + \frac{.09}{4}\right)^{4t}$$

$$2 = (1.0225)^{4t}$$

$$\log 2 = \log(1.0225)^{4t}$$

$$\log 2 = 4t \log(1.0225)$$

$$\frac{\log 2}{4 \log 1.0225} = t$$

$$\frac{.30103}{.0386533} = t \qquad 7.79 \text{ years} = t$$

20. $\cot^2\theta + 2\sin\theta = \csc^2\theta - 2$

$$\csc^2\theta - 1 + 2\sin\theta = \csc^2\theta - 2$$

$$\csc^2\theta - 1 - \csc^2\theta + 2\sin\theta = -2$$

$$2\sin\theta = -2 + 1$$

$$\sin\theta = {}^-\tfrac{1}{2}$$

$$\theta \to \text{III}, \text{IV}$$

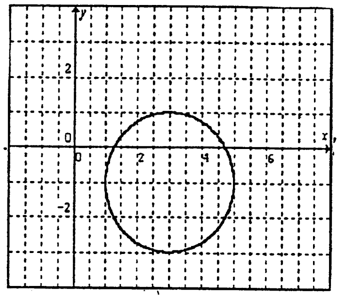
$$\theta = \frac{7\pi}{6} + 2k\pi, \ \frac{11\pi}{6} + 2k\pi, \ k \text{ is an integer}$$

Exercise 37: Circle and Ellipse

1. $C=(3,-1)$ $R=2$

Eqn: $(x-3)^2+(y+1)^2=4$

$x^2+y^2-6x+2y+6=0$



2. $C=(0,0)$

$R=\sqrt{(0-4)^2+(0+5)^2}=\sqrt{41}$

$x^2+y^2-41=0$

3. centre = midpoint $AB=\left(\frac{6-2}{2},\frac{-8+4}{2}\right)=(2,-2)$

radius $=\frac{\text{length }AB}{2}=\frac{\sqrt{(6+2)^2+(-8-4)^2}}{2}=\frac{\sqrt{208}}{2}=\frac{2\sqrt{52}}{2}=\sqrt{52}$

Eqn: $(x-2)^2+(y+2)^2=52$

$x^2+y^2-4x+4y-44=0$

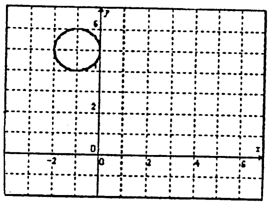
4. a) $(x^2+2x)+(y^2-10y)=-25$

$(x^2+2x+1)+(y^2-10y+25)=-25+1+25$

$(x+1)^2+(y-5)^2=1$

Centre: $(-1,5)$

radius: 1



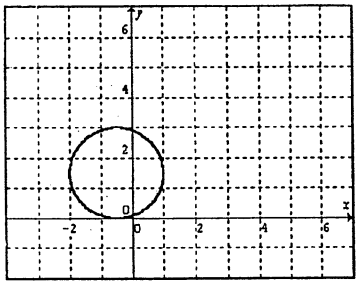
b) $4x^2+4x+4y^2-12y=-1$

$4(x^2+x+\frac{1}{4})+4(y^2-3y+\frac{9}{4})=-1+1+9$

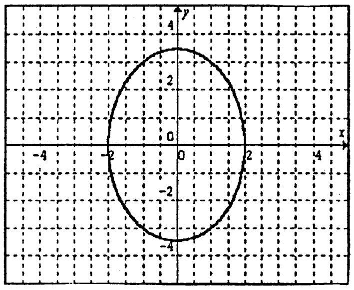
$4(x+\frac{1}{2})^2+4(y-\frac{3}{2})^2=9$

$(x+\frac{1}{2})^2+(y-\frac{3}{2})^2=\frac{9}{4}$

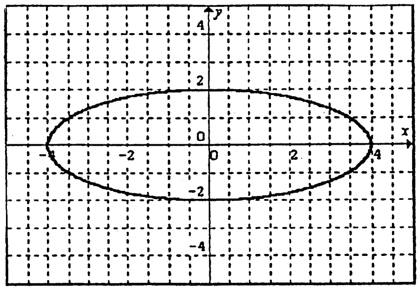
$C:(-\frac{1}{2},\frac{3}{2})$ $r=\frac{3}{2}$



5. a)

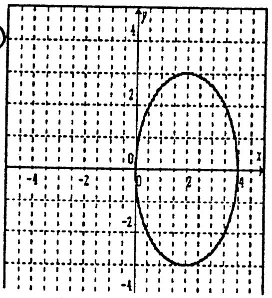


b)

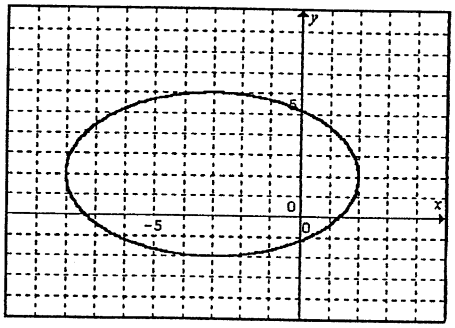


Exercise 37: Circle and Ellipse (continued)

6. a)



b)



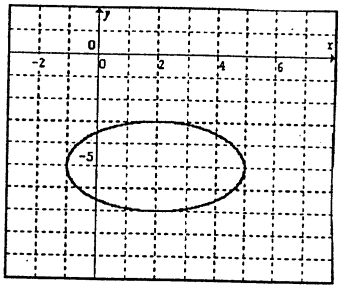
7. a) $4x^2 - 16x + 9y^2 + 90y = -205$

$4(x^2 - 4x + 4) + 9(y^2 + 10y + 25) = -205 + 16 + 225$

$\frac{4(x-2)^2}{36} + \frac{9(y+5)^2}{36} = \frac{36}{36}$

$\frac{(x-2)^2}{9} + \frac{(y+5)^2}{4} = 1$

$C : (2,-5)$



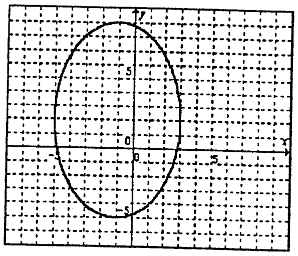
b) $49x^2 + 98x + 16y^2 - 64y = 671$

$49(x^2 + 2x + 1) + 16(y^2 - 4y + 4) = 671 + 49 + 64$

$\frac{49(x+1)^2}{784} + \frac{16(y-2)^2}{784} = \frac{784}{784}$

$\frac{(x+1)^2}{16} + \frac{(y-2)^2}{49} = 1$

$C : (-1, 2)$



8. a) Centre $(0,0)$

$a = 6$

$b = 3$

$\frac{x^2}{36} + \frac{y^2}{9} = 1$

b) Centre = midpoint $AB = \left(\frac{-4+8}{2}, \frac{3+3}{2}\right) = (2,3)$

$a = \frac{1}{2}[8-(-4)] = 6$

$b = \frac{1}{2}[5-1] = 2$

$\frac{(x-2)^2}{36} + \frac{(y-3)^2}{4} = 1$

9. Replace b^2 with a^2

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1$ is equivalent to $(x-h)^2 + (y-k)^2 = a^2$

This is a circle with centre (h,k) and radius a.

Exercise 37: Circle and Ellipse (continued)

10. Centre is $\left(\frac{-4+10}{2}, 2\right) = (3,2)$

$a = \sqrt{(10-3)^2+(2-2)^2} = 7$

$b = \frac{1}{2}(10) = 5$

Equation: $\frac{(x-3)^2}{49} + \frac{(y-2)^2}{25} = 1$

11. Let us find how many diagonals we can draw. To find a diagonal we must have two endpoints. The first point can be selected in n ways. The second point must not be the same as the first, and must not be adjacent to it on either side. Thus we may select the second point in $n-3$ ways. This seems as if it gives $n(n-3)$ diagonals. However, we have counted $\overline{AB}$ and $\overline{BA}$ separately, so we must divide by 2 to eliminate this overcount.

12. $\dfrac{\dfrac{2n(2n-1)(2n-2)}{3!}}{\dfrac{n(n-1)}{2!}} = \dfrac{44}{3}$ or $\dfrac{2(2n-1)(2)}{3} = \dfrac{44}{3}$

$8n-4=44$ so $n=6$

13. $\sec^2 x \ \csc^2 x = \sec^2 x + \csc^2 x$

R.H.S. $\dfrac{1}{\cos^2 x} + \dfrac{1}{\sin^2 x}$

$= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x \ \sin^2 x}$

$= \dfrac{1}{\cos^2 x \ \sin^2 x}$

$= \dfrac{1}{\cos^2 x} \cdot \dfrac{1}{\sin^2 x}$

$= \sec^2 x \ \csc^2 x$

$=$ R.H.S.

14. $b^{x^2+x} = b^0$

$x^2+x=0$

$x(x+1)=0 \quad x=0,-1$

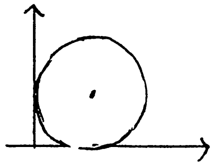
Exercise 37: Circle and Ellipse (continued)

15. Case One: Let us start with a 2. The remaining 6 numbers, three of which are alike, can be arranged in(6!)/(3!) = 120 ways.

Case Two: Let us begin with a 3. The remaining 6 numbers contain two 2's and two 3's. They can be arranged in (6!)/[(2!)(2!)] = 180 ways.

Case Three: Let us begin with a 4. The remaining 6 numbers contain two 2's and three 3's. They can be arranged in (6!)/[(2!)(3!)] = 60 ways.
Thus, there are 120 + 180 + 60 = 360 ways.

16. $\pi r^2 = 25\pi$
$R = 5$
$C = (5,5)$
Eqn: $(x-5)^2+(y-5)^2 = 25$
$x^2+y^2-10x-10y+25=0$



17. $\frac{\log 16}{\log 5} = 1.72271$

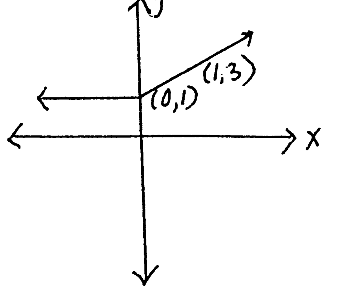
18. $\log_3 \frac{5}{2} = \log_3 5 - \log_3 2$

$= 1.465 - .63 = .835$

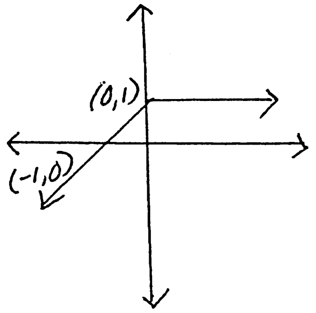
19. $\sec \theta/2 = -2.9413$
$\cos \theta/2 = -.3400$
related angle $= 70.1°$

$\theta/2 = 109.9°$
$\theta = 219.8°$

20. a)

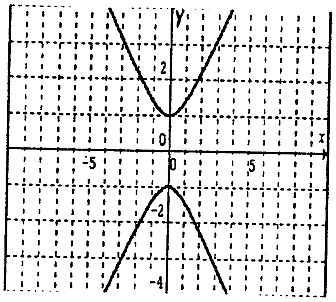


b)

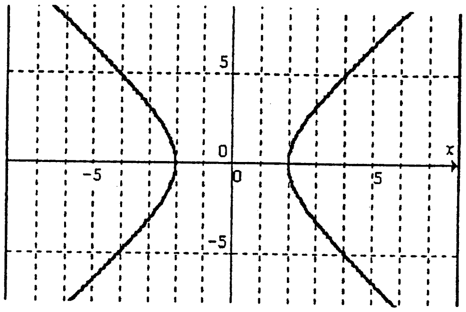


Exercise 38: Hyperbola

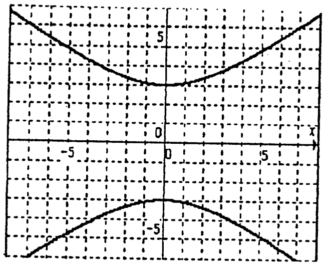
1. a)



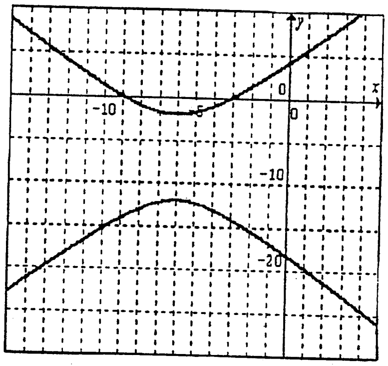
b)



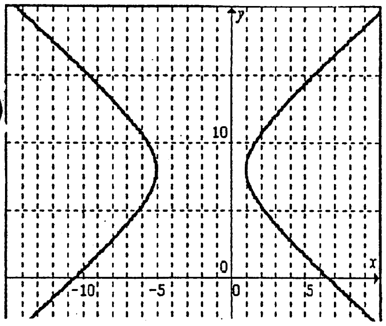
2. a



b)



3. a)



$(x^2+4x \quad) - (y^2-16y \quad) = 69$

$(x^2+4x+4) - (y^2-16y+64) = 9$

$(x+2)^2 - (y-8)^2 = 9$

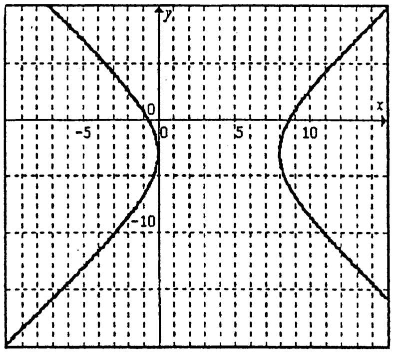
centre $(-2,8)$ $\quad \dfrac{(x+2)^2}{9} - \dfrac{(y-8)^2}{9} = 1$

b) $(25x^2-200x) - (16y^2+96y) = 144$

$25(x^2-8x+16) - 16(y^2+6y+9) = 144+400-144$

$25(x-4)^2 - 16(y+3)^2 = 400$

$\dfrac{(x-4)^2}{9} - \dfrac{(y+3)^2}{25} = 1$



Exercise 38: Hyperbola (continued)

4. $a = 4$ $\frac{b}{4} = \frac{3}{2}$ $2b = 12$
$b = 6$

$$\frac{x^2}{16} - \frac{y^2}{36} = 1$$

5. Since one vertex is at $(2, -4)$ this tells us that it opens up and down. This means the denominator of the y term is $(5)^2$. [Slope is $5/7$ so $5/a = 5/7$ $\Rightarrow a = 7$.]

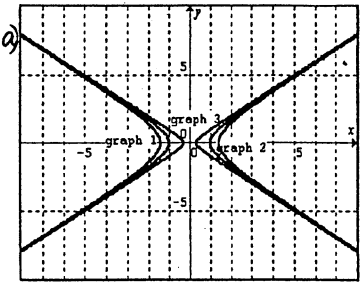
slope $\rightarrow$ $5x - 7y = 3$
$-7y = -5x + 3$
$y = \frac{5}{7}x + \frac{3}{7}$

$$\frac{(y-1)^2}{25} - \frac{(x-2)^2}{49} = 1$$

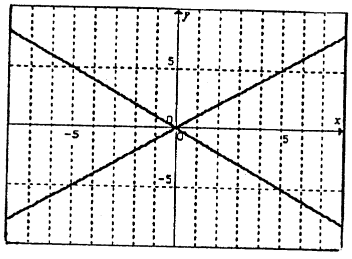
6. a) Ellipse stretched along y-axis
b) Hyperbola opening along y-axis
c) Parabola opening to positive x-axis
d) Hyperbola opening along x-axis
e) Parabola opening to negative y-axis
f) Ellipse stretched along y-axis

7. a) ellipse b) hyperbola c) circle d) parabola

8. a)



b)



The graph of $x^2 - y^2 = 0$ consists of 2 intersecting lines. (These lines are the asymptotes to the graph in Part A.)

Exercise 38: Hyperbola (continued)

9. Since $a = 4$ and $b = 3$ one vertex of the rectangle is $(4,3)$. This point is $\sqrt{16+9} = 5$ units from the origin. The equation of the circle is $x^2 + y^2 = 25$.

10. A is the point $(10, \sqrt{10K})$

$AB = 2\sqrt{10K}$

Area of $\Delta AOB = 40$

$\frac{1}{2}(2\sqrt{10K})(10) = 40$

$\sqrt{10K} = 4$

$10K = 16$

$K = 1.6$

OR

Let $AB = b$

$\frac{1}{2} b(10) = 40$

$b = 8$

$\therefore$ A is $(10, 4)$

$y^2 = Kx$

$16 = k(10)$

$K = 1.6$

11. $3^{\log_3 4 + \log_3 5} = 3^{\log_3 20} = 20$

12. $\ln(x+5) + \ln 5 = \ln 65$

$(x+5)5 = 65$

$5x + 25 = 65$

$5x = 40$

$x = 8$

13. $10^4 = 10\,000$

14. $\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} = \dfrac{1}{\sin^2\theta - \sin^4\theta}$

$$\text{L.H.S} = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\ \sin^2\theta}$$

$$= \frac{1}{\cos^2\theta\ \sin^2\theta}$$

$$= \frac{1}{(1-\sin^2\theta)(\sin^2\theta)}$$

$$\frac{1}{\sin^2\theta - \sin^4\theta}$$

Exercise 38: Hyperbola (continued)

15. If there were no restrictions the 10 papers could be arranged in 10! ways.
If the best and the worst were together, there would be 2 ways of putting them together. Considering them as one object, the 9 objects could then be arranged in 9! ways. There are thus (2)(9!) ways in which these papers are together.

There will then be $10! - 2(9!) = 3628800 - (2)\,362880 = 2903040$ ways in which these papers are not together.

solution 1:

16. $\log e = \frac{1}{\ln 10}$

L.H.S.

$$\log_{10} e = \frac{\log e}{\log 10}$$

$$= \frac{\log_{10} e}{\log_{10} 10}$$

$$= \frac{\log_{10} e}{1}$$

$x = \log_{10} e$

$10^x = e$

$\ln 10^x = \ln e$

$x \ln 10 = 1$

$$x = \frac{1}{\ln 10}$$

$$\therefore \log e = \frac{1}{\ln 10}$$

solution 2: $\log e = \frac{\log_e e}{\log_e 10}$

$$\log e = \frac{1}{\ln 10}$$

17. $\log_5 (x+1) - \log_5 8 = \log_5 (x-3) - \log_5 6$

$\log_5 (x+1) - \log_5 (x-3) = \log_5 8 - \log_5 6$

$\log_5 \frac{(x+1)}{x-3} = \log_5 \frac{8}{6}$

$\frac{x+1}{x-3} = \frac{8}{6}$

$6x + 6 = 8x - 24$

$30 = 2x$

$x = 15$

check:

LHS. $\log_5 (15+1) - \log_5 8$

$= \log_5 16 - \log_5 8$

$= \log_5 \frac{16}{8} = \log_5 2$

R.H.S: $\log_5 (15-3) - \log_5 6$

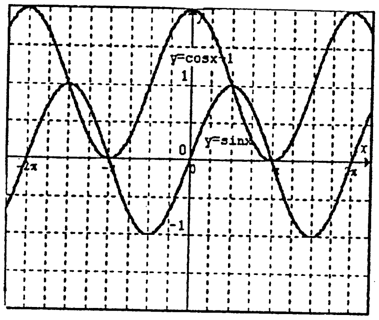
$\log_5 12 - \log_5 6$

$\log_5 \left(\frac{12}{6}\right) = \log_5 2$

$\therefore$ LHS = RHS.

Exercise 38: Hyperbola (continued)

18.



a) 1.571, 3.142

b) $\frac{\pi}{2}$, π

check $\frac{\pi}{2}$

$\cos\frac{\pi}{2}+1$ $\quad$ $\sin\frac{\pi}{2}$

$=1$ $\quad$ $=1$

$\therefore$ LHS = RHS

check π

$\cos\pi+1$ $\quad$ $\sin\pi$

$=0$ $\quad$ $=0$

$\therefore$ LHS = RHS

19. $3(\sec^2\theta-1)=7\sec\theta-5$

$3\sec^2\theta-3=7\sec\theta-5$

$3\sec^2\theta-7\sec\theta+2=0$

$(3\sec\theta-1)(\sec\theta-2)=0$

$\sec\theta=\frac{1}{3}$ $\quad$ $\sec\theta=2$

$\cos\theta=3$ $\quad$ $\cos\theta=\frac{1}{2}$

θ is undefined

$\theta=60°, 300°$

20. First term has x^{36}; second term has $(x^4)^8\cdot\frac{1}{x}=x^{31}$; exponent decreases by 5 7th term is:

${}_9C_6(2x^4)^3(-\frac{1}{2x})^6=\frac{9!}{3!6!}(8x^{12})(\frac{1}{64x^6})$

$=\frac{21x^6}{2}$

or $(x^4)^{9-r}(\frac{1}{x})^r=x^6$

$x^{36-4r}\cdot x^{-r}=x^6$

$36-5r=6$

$-5r=6$

$-5r=-30$

$r=6$ $\therefore$ 7th term

${}_9C_6(2x^4)^{9-6}(-\frac{1}{2x})^6=\frac{21x^6}{2}$

Exercise 39: Sample Spaces

1. a) H, T b) 1, 2, 3, 4, 5, 6 c) (H, H), (H, T), (T, H), (T, T)
d) (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)
e) (H, H, H), (H, H, T), (H, T, H), (T, H, H), (T, T, T), (T, T, H), (T, H, T), (H, T, T)

2. a) (5, 7), (5, 8), (5, 9)
(6, 7) (6, 8) (6, 9)
(7, 7) (7, 8) (7, 9)
(8, 7) (8, 8) (8, 9)

b) (3)(4) = 12

c) 2 points, viz (7, 7)(8, 8)

d) $\frac{1}{12}$

	Bus 5	6	7	8
Train 7	5, 7	6, 7	7, 7	8, 7
8	5, 8	6, 8	7, 8	8, 8
9	5, 9	6, 9	7, 9	8, 9

3. a) Independent b) Dependent c) Independent
d) Independent

4. 1 - P

5. a) $\frac{4}{52}$ b) $\frac{13}{52}$ c) $\frac{1}{52}$

6. a) $\frac{3}{10}$ b) $\frac{7}{10}$ c) 1 (certainty)

7. a) $\frac{6}{36}$ b) $\frac{10}{36}$ 8. $\frac{16}{52}$

9. $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = 2\csc^2 x$

L.H.S. $\frac{1}{1+\cos x} + \frac{1}{1-\cos x}$

$$= \frac{1-\cos x + 1 + \cos x}{(1+\cos x)(1-\cos x)}$$

$$= \frac{2}{1-\cos^2 x}$$

$$= \frac{2}{\sin^2 x}$$

$$= 2\csc^2 x = R.H.S$$

10. $\log_3 x^2 + \log_3 x^3 = \log_3 16x$

$\log_3 (x^2)(x^3) = \log_3 16x$

$\log_3 x^5 = \log_3 16x$

$x^5 = 16x$

$x^5 - 16x = 0$

$x(x^4 - 16) = 0$

$x = 0$ or $x = 2$ or $x = -2$

check: $\log_3 x^2 + \log_3 x^3 = \log_3 16x$

$x = 0 \Rightarrow \log_3 0^2 + \log_3 0^3 = \log_3 16(0)$ $\therefore$ reject

$x = 2 \Rightarrow \log_3 4 + \log_3 8 = \log_3 32$ ✓

$x = -2 \Rightarrow \log_3 4 + \log_3 (-8)$ $\therefore$ reject $\therefore x = 2$

Exercise 39: Sample Spaces (continued)

11. $2^{x+3}=3^{x-1}$

$(x+3)\log 2=(x-1)\log 3$

$x\log 2+3\log 2=x\log 3-\log 3$

$x\log 2-x\log 3=-\log 3-3\log 2$

$x(\log 2-\log 3)=-\log 3-3\log 2$

$x=\dfrac{-\log 3-3\log 2}{\log 2-\log 3}$

$x=7.838$

12. $\log_b 108=\log_b(27\cdot 4)$

$=\log_b(3^3\cdot 4)$

$=\log_b 3^3+\log_b 4$

$=3\log_b 3+\log_b 4$

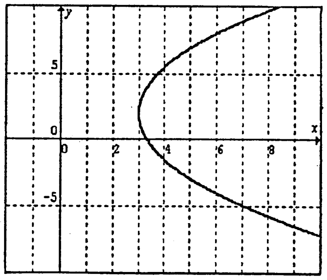
$=3(0.613)+0.774$

$=2.613$

13. a) $p^{10}+10p^9q+45p^8q^2$ and $45p^2q^8+10pq^9+q^{10}$

b) 4th term $_{10}C_7=120$ c) 6th term d) 11 terms

14.



$y^2-4y-12x+40=0$

$(y-2)^2-4-12x+40=0$

$(y-2)^2=12x-36$

$(y-2)=12(x-3)$

15. a) parabola with vertical axis of symmetry

b) parabola with horizontal axis of symmetry

c) hyperbola

d) circle

e) oblique line

f) ellipse

g) horizontal line

h) vertical line

Exercise 39: Sample Spaces (continued)

16. a) $A = 20e^{-Kt}$

$14 = 20e^{-K(10)}$

$0.7 = e^{-K(10)}$

$\ln(0.7) = -10K$

$\dfrac{\ln(0.7)}{-10} = K$

$\therefore 10 = 20e^{\frac{\ln(0.7)}{-10} \cdot t}$

$\ln(5) = \dfrac{\ln(0.7)}{-10} \cdot t$

$t = \dfrac{\ln(5)}{\left(\dfrac{\ln(0.7)}{-10}\right)} = 19.43$ days

b) $A = 20e^{-Kt}$

$A = 20e^{-0.0357(17)}$

$A = 10.9$ g

17. a) $\theta_r = 0.432$

$\underline{\text{III}} : \pi + \theta r = 3.57$

$\underline{\text{IV}} : 2\pi - \theta r = 5.85$

$\therefore \{3.57, 5.85\}$

b) $\theta_r = 1.06$

$\underline{\text{II}} : \pi - 1.06 = 2.08$

$\underline{\text{IV}} : 2\pi - 1.06 = 5.22$

$\therefore \{2.08, 5.22\}$

18. ${}_{11}C_6 \cdot 5! \cdot 4! = 1\,330\,560$

19. $9\csc^2\theta - 9 = 4\csc^2\theta$

$5\csc^2\theta = 9$

$\csc^2\theta = \dfrac{9}{5}$

$\sin^2\theta = \dfrac{5}{9}$

$\sin\theta = \dfrac{\pm\sqrt{5}}{3}$

$\theta = 0.8411 + 2k\pi$

or

$\theta = \pi - 0.8411 = 2.3005 + 2k\pi$

or

$\theta = -0.8411 + 2\pi = 5.4421 + 2k\pi$

or

$\theta = \pi - (-0.8411) = 3.9827 + 2k\pi$

20. $A = P\left(1 + \frac{r}{s}\right)^{ns}$

$A = 1200\left(1 + \frac{0.06}{2}\right)^{12.2}$

$A = \$\,2439.35$

Exercise 40: Probability of Independent and Dependent Events

1\. a) $P(Ace, Ace) = \left(\frac{4}{52}\right) \cdot \left(\frac{4}{52}\right) = \frac{1}{169}$ first card is replaced

b) $P(Ace, Ace) = \left(\frac{4}{52}\right) \cdot \left(\frac{3}{51}\right) = \frac{1}{221}$ first card not replaced

2\. $P(4, \text{ or } 5, \text{ or } 6, 1 \text{ or } 2 \text{ or } 3 \text{ or } 4) = \frac{3}{6} \cdot \frac{4}{6} = \frac{1}{3}$

3\. a) $P(white, white) = \left(\frac{4}{6}\right) \cdot \left(\frac{3}{8}\right) = \frac{1}{4}$

b) $P(black, black) = \left(\frac{2}{6}\right)\left(\frac{5}{8}\right) = \frac{5}{24}$

c) $P(black, white) + P(white, black) = \frac{2}{6} \times \frac{3}{8} + \frac{4}{6} \times \frac{5}{8} = \frac{13}{24}$

4\. a) $P(Red, Red) = \frac{26}{52} \times \frac{26}{52} = \frac{1}{4}$

b) $P(Heart, Heart) = \frac{13}{52} \times \frac{13}{52} = \frac{1}{16}$

5\. $P(\text{ODD sum}) = P(odd, even) + P(even, odd)$

$= \frac{5}{10} \times \frac{5}{9} + \frac{5}{10} \times \frac{5}{9}$

$= \frac{5}{9}$

6\. $P(tails, tails, tails) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$P(\text{not all tails}) = 1 - P(\text{all tails})$

$= 1 - \frac{1}{8}$

$= \frac{7}{8}$

7\. $P = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6}$

Exercise 40: Probability of Independent and Dependent Events (continued)

8. $_7C_0(2x^3)^7 \Rightarrow 128x^{21}$

$_7C_1(2x^3)^6(-3y^2)^1 \Rightarrow 7 \cdot 64x^{18} \cdot (-3)y^2 \Rightarrow -1344x^{18}y^2$

$_7C_2(2x^3)^5(-3y^2)^2 \Rightarrow \frac{7!}{5!2!} \quad 2^5x^{15}(-3)^2y^4 \Rightarrow 21 \cdot 32 \cdot 9x^{15}y^4$

$_7C_3(2x^3)^4(-3y^2)^3 \Rightarrow \frac{7!}{4!3!} \quad 2^4x^{12}(-3)^3y^6 \quad \boxed{\Rightarrow 6048x^{15}y^4}$

$\Rightarrow -15120x^{12}y^6$

9. $P(\text{Red or White}) = \frac{2}{9} + \frac{3}{9} = \frac{5}{9}$

10. $P(\text{yellow, yellow or black, black}) = \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11} = \frac{31}{66}$

11. (1,1) (1,2) (1,3) (1,4)
(2,1) (2,2) (2,3) (2,4)
(3,1) (3,2) (3,3) (3,4)
(4,1) (4,2) (4,3) (4,4)

12. Answers will vary.

13. $P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$

14. $P(\text{face card}) = \frac{12}{52} = \frac{3}{13}$

15. $x^2 - 8x + y^2 - 4y + 19 = 0$

$(x-4)^2 - 16 + (y-2)^2 - 4 + 19 = 0$

$(x-4)^2 + (y-2)^2 = 1$

$R = 1 \quad \text{Area} = \pi r^2 = 3.14 \text{ units}^2$

16. $4^{2x} = 2^{x(x-2)}$

$((2)^2)^{2x} = 2^{x(x-2)}$

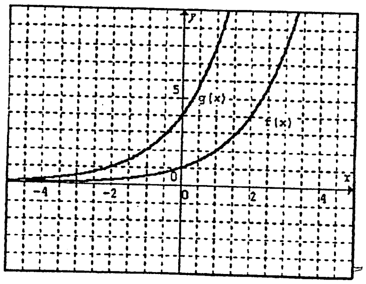
$4x = x^2 - 2x$

$0 = x^2 - 6x$

$0 = x(x-6)$

$\therefore x = 0 \text{ or } 6$

17.



$f(x) = 2^x$

$g(x) = 4 \cdot 2^x$

$= 2^2 \cdot 2^x$

$= 2^{x+2}$

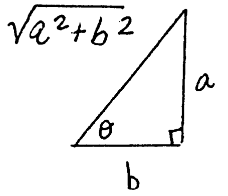
shifts the graph of f(x) 2 units to the left or a vertical stretch by a factor of 4

	f(x)	g(x)
Horizontal Asymptote	y = 0	y = 0
Domain:	Reals	Reals
Range:	{y | y > 0}	{y | y > 0}
X-intercept	∅	∅
y-intercept	1	4
Increasing	✓	✓

Exercise 40: Probability of Independent and Dependent Events (continued)

18. If $0 < \theta \frac{\pi}{2}$ and $\tan\theta = \frac{a}{b}$

then



the hypotenuse is $\sqrt{a^2+b^2}$

$\therefore \sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

Alternate Solution #1

$$\sin\theta = \frac{1}{\csc\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+\frac{b^2}{a^2}}} = \frac{1}{\sqrt{\frac{a^2+b^2}{a^2}}} = \frac{a}{\sqrt{a^2+b^2}}$$

Alternate Solution #2

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = 1-\cos^2\theta$$
$$\sin\theta = \sqrt{1-\cos^2\theta}$$
$$= \sqrt{1-\frac{b^2}{a^2+b^2}}$$
$$= \sqrt{\frac{a^2+b^2-b^2}{a^2+b^2}}$$
$$= \sqrt{\frac{a^2}{a^2+b^2}}$$
$$= \frac{a}{\sqrt{a^2+b^2}}$$

20. $e^{\ln 3 - \ln 2} = x$

$$\ln e^{\ln 3 - \ln 2} = \ln x$$
$$\ln 3 - \ln 2 = \ln x$$
$$\ln \frac{3}{2} = \ln x$$
$$\frac{3}{2} = x$$

19. $\left(\frac{2}{3}x - \frac{3}{2x^2}\right)^{10} \Rightarrow 11$ terms

middle term is sixth term

$${}_{10}C_5 \left(\frac{2x}{3}\right)^5 \left(\frac{-3}{2x^2}\right)^5$$
$$\frac{10!}{5!\,5!} \; \frac{2^5x^5}{3^5} \; \frac{(-3)^5}{(2^5)x^{10}}$$
$$\frac{252 \cdot 32 \cdot (-243)}{(243)(32)x^5}$$
$$-\frac{252}{x^5}$$

seventh term is

$${}_{10}C_6 \left(\frac{2x}{3}\right)^4 \left(\frac{-3}{2x^2}\right)^6$$
$$\frac{10! \; 2^4 \; x^4(-3)^6}{6!\,4! \; 3^4 \; 2^6 \; x^{12}}$$
$$\frac{210 \cdot 9}{4x^8}$$
$$\frac{945}{2x^8}$$

Exercise 41: Combining Probabilities

1. $P(\text{losing}) = 1 - P(\text{winning})$

$= 1 - \frac{1}{31}$

$= \frac{30}{31}$

2. a) $P(\text{Not } A + B) = 0.7 \times 0.4$

$= 0.28$

b) $P(\text{Not} A + \text{Not} B + \text{Not} A + \text{Not } B + A) = 0.7 \times 0.6 \times 0.7 \times 0.6 \times 0.3$

$= 0.05292$

3. $P(\text{face}) = \frac{12}{52} = \frac{3}{13}$

4. Assumption: 6 - sided die $(6,6)(6,5)(6,4)(4,6)(5,6)(5,5)$

$P(\text{sum greater than } 9) = \frac{6}{36} = \frac{1}{6}$

5. $P(\text{red}, \text{red}) = \frac{4}{11} \times \frac{3}{10}$

$= \frac{6}{55}$.

6. $P(K \text{ or Red}) = P(K) + P(R) - P(K \text{ and Red})$

$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$

$= \frac{28}{52}$

$= \frac{7}{13}$

7. $P(5, H) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

8. $P(G, G) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

9. $P(\text{Red}, \text{Black}) = \frac{26}{52} \cdot \frac{26}{51}$

$= \frac{13}{51}$

Exercise 41: Combining Probabilities (continued)

10. a) P(Defective, Defective, Defective) $= \frac{1}{200} \times \frac{1}{200} \times \frac{1}{200}$

$= \frac{1}{8000000}$

b) P(Good, Good, Good) =

$\frac{199}{200} \cdot \frac{199}{200} \cdot \frac{199}{200} =$

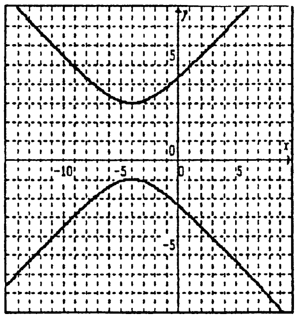
$\frac{788\,0599}{8000000}$

11. $4(x^2+8x)-9(y^2-2y)=-91$

$4(x^2+8x+16)-9(y^2-2y+1)=-91+64-9$

$4(x+14)^2-9(y-1)^2=-36$

$\frac{(y-1)^2}{4}-\frac{(x+4)^2}{9}=1$



12. independent

13. a) P(odd) $= \frac{3}{6} = \frac{1}{2}$

b) P(ace or ten of diamonds or two of spades) $= \frac{6}{52}$ or $\frac{3}{26}$

c) P(tails) = ½ (regardless of previous outcomes)

14. $\tan^2\theta = \sec\theta + 1$

$\frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos\theta} + 1$

$\sin^2\theta = \cos\theta + \cos^2\theta$ $\quad \cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$1-\cos^2\theta = \cos\theta + \cos^2\theta$

$2\cos^2\theta + \cos\theta - 1 = 0$ $\quad \cos\theta = -1 \quad \theta = \pi$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

15 a) $x^2+y^2=4$

b) $y = x^2-2x+3$

c) $x^2-y^2=1$

d) $3x^2+2y^2=6$

e) $y=\sqrt{x}$

f) $y-2=0$

g) $x=3$

h) $4x-2y+5=0$

i) $y=|x|$

16. $\log_4 x = 0$

$4^0 = x$

$x = 1$

Exercise 41: Combining Probabilities (continued)

17. $\frac{\cos x}{\csc x} - \frac{\sin x}{\tan x} = \frac{\sin x - 1}{\sec x}$

$\frac{\frac{\cos x}{1}}{\frac{1}{\sin x}} - \frac{\sin x}{\frac{\sin x}{\cos x}}$ | $\frac{\sin x - 1}{\frac{1}{\cos x}}$

$\sin x \cos x - \cos x$ | $\cos x(\sin x - 1)$

$\cos x(\sin x - 1)$

LHS = RHS

18. $A_{\text{Big Circle}} = 196\pi$ Assumption: Radius of each circle is 7 units.

$\therefore$ Area of small cirles:

area $= 2 \times 49\pi$

$= 98\pi$

Shaded area $= 196\pi - 98\pi$

$= 98\pi$

$= 307.88$ units^2

19. Term 1: x^{70}

Term 2: $\frac{x^{63}}{x^4} = x^{59}$

Term 3: $\frac{x^{56}}{x^8} = x^{48}$

Term 4: x^{37}

Term 5: x^{26}

Term 6: x^{15}

Term 7: x^4

Term 8: $\frac{1}{x^7}$

$\therefore$ Term 8

$_{10}C_7\left(\frac{2x^7}{3}\right)^3\left(\frac{-3}{2x^7}\right)^7$

$\frac{10!\ 2^3\ x^2\ (-3)^7}{7!\ 3!\ 3^3\ 2^7 x^{28}}$

$-\frac{120 \cdot (3)^4}{2^4\ x^7}$

$\frac{9720}{16x^7}$

$\frac{1215}{2x^7}$

20. solution 1:

$500 = 11^{\log_{11} 50}$

$11^{2.5917}$

solution 2: $11^x = 50$

$x = \log_{11} 50$

$x = \frac{\log 500}{\log 11}$

$x = 2.5917$

$11^{2.5917}$

solution 3:

$11^x = 50$

$x \log 11 = \log 50$

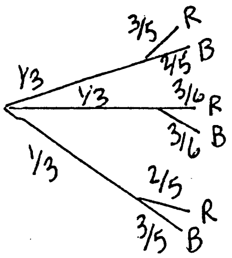
$x = \frac{\log 50}{\log 11}$

$x = 2.5917$

$11^{2.5919} = 50$

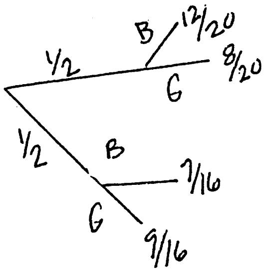
Exercise 42: Conditional Probability I

1.



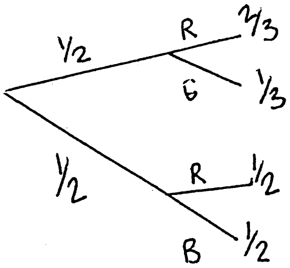
$$\begin{aligned} P(R) &= \frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{3}{6} + \frac{1}{3} \times \frac{2}{5} \\ &= \frac{1}{5} + \frac{1}{6} + \frac{2}{15} \\ &= \frac{6}{30} + \frac{5}{30} + \frac{4}{30} \\ &= \frac{15}{30} = \frac{1}{2} \end{aligned}$$

2.



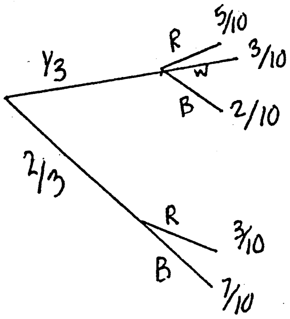
$$\begin{aligned} P(G) &= \frac{1}{2} \times \frac{8}{20} + \frac{1}{2} \times \frac{9}{16} \\ &= \frac{4}{20} + \frac{9}{32} \\ &= \frac{32}{160} + \frac{45}{160} \\ &= \frac{77}{160} \end{aligned}$$

3.



$$\begin{aligned} P(R) &= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{3} + \frac{1}{4} \\ &= \frac{7}{12} \end{aligned}$$

4.

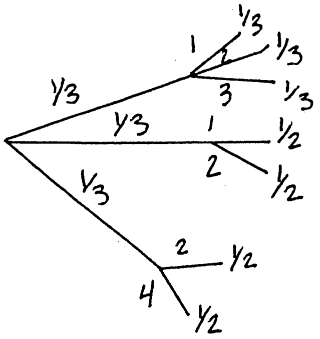


$$P(R) = \frac{1}{3} \times \frac{5}{10} + \frac{2}{3} \times \frac{3}{10} = \frac{11}{30}$$

$$P(W) = \frac{1}{3} \times \frac{3}{10} = \frac{1}{10}$$

$$P(B) = \frac{1}{3} \times \frac{2}{10} + \frac{2}{3} \times \frac{7}{10} = \frac{16}{30} = \frac{8}{15}$$

5.



$$\begin{aligned} P(2) &= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} \\ &= \frac{1}{9} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{8}{18} \\ &= \frac{4}{9} \end{aligned}$$

Exercise 42: Conditional Probability I (continued)

6. a) $P(\text{Even}) = P(2) + P(4)$

$= 4/9 + 1/3 \times 1/2$

$= 4/9 + 1/6$

$= 11/18$

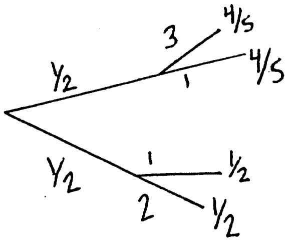
b) $P(<3) = P(1) + P(2)$

$= (1/3 \times 1/3 + 1/3 \times 1/2) + 4/9$

$= 1/9 + 1/6 + 4/9$

$= 13/18$

7.



$P(1) = 1/2 \times 4/5 + 1/2 \times 1/2$

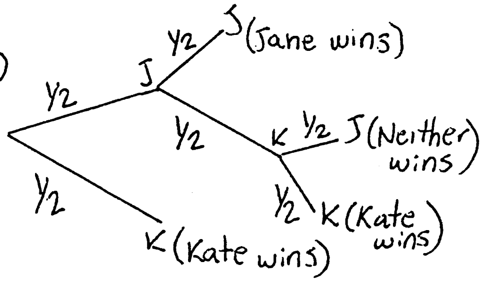
$= 2/5 + 1/4$

$= 13/20$

$P(2) = 1/2 \times 1/2 = 1/4$

$P(3) = 1/2 \times 1/5 = 1/10$

8. a)



b) $P(J) = 1/2 \times 1/2 = 1/4$

c) $P(K) = 1/2 + 1/2 \times 1/2 \times 1/2 = 5/8$

d) $P(\text{Neither}) = 1/2 \times 1/2 \times 1/2 = 1/8$

9. a) Choose the face value of the 4 of a kind in 13 ways. Choose the single card in 48 ways. ∴ $13 \times 48 = 624$ ways.

b) Choose the face value of the 3 of a kind in 13 ways. Choose the 3 of a kind in ${}_4C_3 = 4$ ways. Choose the face values of the 2 different cards in ${}_{12}C_2 = \frac{12!}{2!\,10!} = 66$ ways. Choose the suit of these cards in 4 ways. ∴ $13 \times 4 \times 66 \times 4 \times 4 = 54912$

10. $7^{x+1} = 343$

$7^{x+1} = 7^3$

$x+1 = 3$

$x = 2$

or $(x+1)\log 7 = \log 343$

$x+1 = \dfrac{\log 343}{\log 7}$

$x = \dfrac{\log 343}{\log 7} - 1$

$x = 2$

Exercise 42: Conditional Probability I (continued)

11. a) (R, R), (R, B), (R, Y), (B, B) (B, R) (B, Y) (Y, Y) (Y, R) (Y, B)

b) no c) Dependent

d) $\frac{11}{18}$

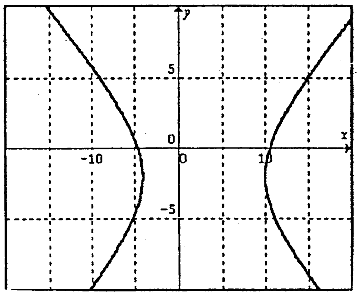
12. $P(H \text{ or } L) = 3/7 + 4/9 = 55/63$

13. $P(M \text{ or } C) = 1/3 + 2/7 = 13/21$

14. $P(\overline{M \text{ or } C}) = 1 - 13/21 = 8/21$

15. a) $P(A \cap B) = 0$ b) $P(A) + P(A) = 1$ c) $P(A \text{ and then } B) = P(A) \times P(B)$ d) $P(A \text{ and the } B) = PA \cdot P(B/A)$

16. $\frac{(x-3)^2}{49} - \frac{(y+2)^2}{25} = 1$



17. $\sec x - \tan x \sin x = \cos x$

Proof:

$\sec x - \tan x \quad \sin x$

$= 1/\cos x - \frac{\sin x}{\cos x} \cdot \frac{\sin x}{1}$

$= \frac{1 - \sin^2 x}{\cos x}$

$= \frac{\cos^2 x}{\cos x}$

$= \cos x \therefore \sec x - \tan x \sin x = \cos x$

18. $\log_x 25 = -2$

$x^{-2} = 25$

$(x^{-2})^{-1/2} = (25)^{-1/2}$

$x = \frac{1}{(25)^{1/2}} \quad x = 1/5$

19. $\log_2(\cos x) = \log_3(1/3)$

$\log_2(\cos x) = -1$

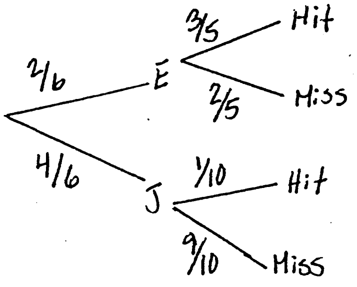
$\cos x = 1/2$

$x = \pi/3, 5\pi/3$

20. The sum of angles in an n-sided polygon is 180(n-2). In a 10 sided polygon this sum is 180(8) = 1440.

Exercise 43: Conditional Probability II

1.

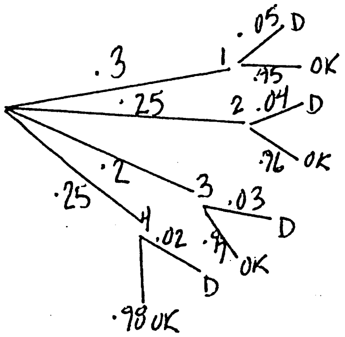


$$P = \frac{P(E,H)}{P(E,H)+P(J,H)}$$

$$= \frac{(2/6)\ (3/5)}{2/6 \times 3/5 + 4/6 \times 1/10}$$

$$= \frac{1/5}{1/5 + 1/15} = \frac{1/5}{4/15} = 3/4$$

2.



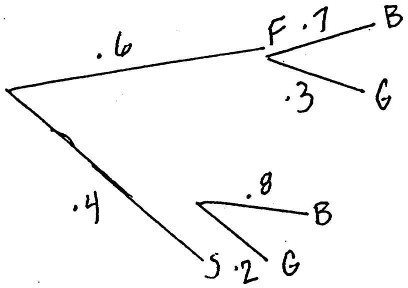
$$P = \frac{P(1,0)}{P(1,0)+P(2,0)+P(3,0)+P(4,0)}$$

$$= \frac{.3 \times .05}{.3 \times .05 + .25 \times .04 + .2 \times .03 + .25 \times .02}$$

$$= \frac{.015}{.015 + .01 + .006 + .005}$$

$$= \frac{.015}{.036} = .42$$

3.



a) $P(G) = P(F,G) + P(S,G)$

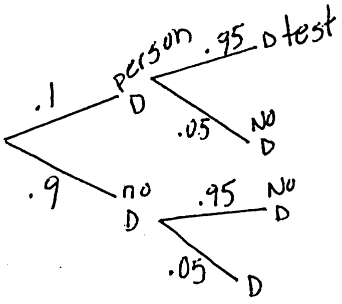
$= .18 + .08$

$= .26$

b) $$P = \frac{P(F,G)}{P(G)}$$

$$= \frac{.18}{.26} = .69$$

4.



$$P = \frac{P(D,D)}{P(D,D) + P(No\ D, D)}$$

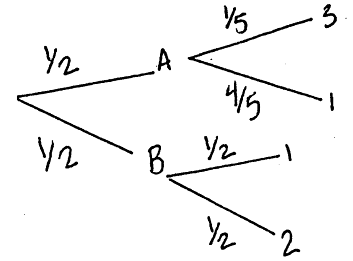
$$= \frac{.1 \times .95}{.1 \times .95 + .9 \times .05}$$

$$= \frac{.095}{.095 + 0045}$$

$$= .68$$

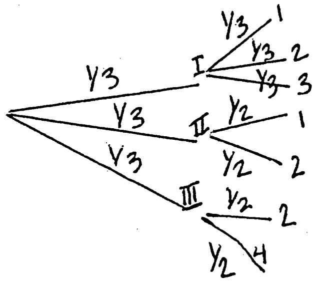
Exercise 43: Conditional Probability II (continued)

5.



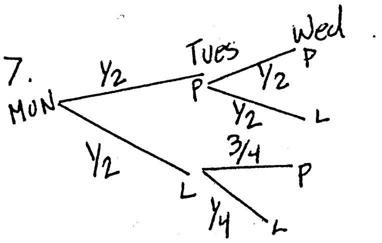
$$P = \frac{P(A,1)}{P(A,1)+P(B,1)}$$
$$= \frac{(1/2)(4/5)}{1/2 \times 4/5 + 1/2 \times 1/2}$$
$$= \frac{2/5}{2/5 + 1/4}$$
$$= \frac{2/5}{13/20} = \frac{40}{65} = \frac{8}{13}$$

6.



$$P = \frac{P(II,2)}{P(I,2)+P(II,2)+P(III,2)}$$
$$= \frac{1/6}{1/9 + 1/6 + 1/6}$$
$$= \frac{1/6}{8/18}$$
$$= 18/48 = 3/8$$

7.



$$P = P(P,P) + P(L,P)$$
$$= 1/2 \times 1/2 + 1/2 \times 3/4$$
$$= 1/4 + 3/8$$
$$= 5/8$$

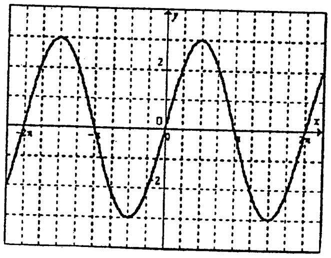
8. $e^{\ln(4x-1)} = 7$

$4x - 1 = 7$

$4x = 8$

$x = 2$

9. $y = 3\cos\left(x - \frac{\pi}{2}\right)$



10. $3^{x(x-4)} = 243$

$3^{x^2-4x} = 3^5$

$x^2 - 4x = 5$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

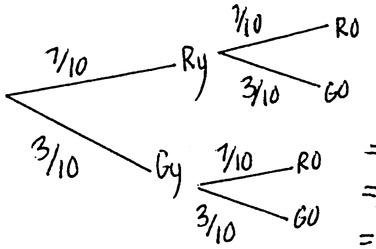
$x = 5, -1$

Exercise 43: Conditional Probability II (continued)

11. $P(G+N) = 2/5 \times 1/3 = 2/15$ 12. $P(\bar{G}+\bar{N}) = 3/5 \times 2/3 = 2/5$

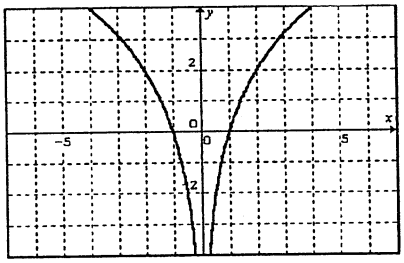
13. $P(A \text{ or } QH) = 4/52 + 1/52 = 5/52$

14.



P(Winning)
$= P(R,R) + P(G,G)$
$= 7/10 \times 7/10 + 3/10 \times 3/10$
$= 49/100 + 9/100$
$= 58/100 = 29/50$

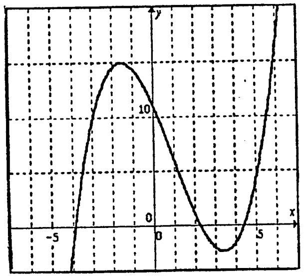
15.



16. $1 - \log(X-4) = \log(x+5)$
$1 = \log(x+5) + \log(x-4)$
$1 = \log(x+5)(x-4)$
$10^1 = (x+5)(x-4)$
$10 = x^2 + x - 20$
$0 = x^2 + x - 30$
$0 = (x+6)(x-5)$
$X = -6 \quad x = 5$
check: extraneous root

17. $\frac{x^2}{9} + \frac{y^2}{25} = 1$
the area is.
$\pi(3)(5) = 15\pi$

18. Shift the original graph 1 left, stretch by a factor of 2 vertically and the shift it 2 down



19. L.H.S
$\sin(\alpha-\beta)\cdot\cos\beta + \cos(\alpha-\beta)\sin\beta$
$= (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\cos\beta + (\cos\alpha\cos\beta + \sin\alpha\sin\beta)\sin\beta$
$= \sin\alpha\cos^2\beta - \cos\alpha\sin\beta\cos\beta + \cos\alpha\cos\beta\sin\beta + \sin\alpha\sin^2\beta$
$= \sin\alpha\cos^2\beta + \sin\alpha\sin^2\beta$
$= \sin\alpha(\cos^2\beta + \sin^2\beta)$
$= \sin\alpha = RHS$

20. $10^{1/2} < 10^{\log_{10} x} < 10^2$
$\sqrt{10} < X < 100$

Exercise 44: Probability Using Permutations and Combinations

1. Sample space = 52^{C_5}

Successes = ${}_4C_4 \times {}_{48}C_1 = 48$

Probability $= \dfrac{48}{2598960} = 1.8 \times 10^{-5}$

2. Sample space = $3!$

Successes. = 1

Probability = $1/6$

3. Sample space = ${}_{18}C_5$

Successes = ${}_{10}C_3 \times {}_8C_2$ $= \dfrac{120 \times 28}{8568}$ $= .39$

Probability $= \dfrac{{}_{10}C_3 \times {}_8C_2}{18\,C_5}$

4. Sample space : (2G3B), (3G2B), (4G1B), (5G0B)

No Combinations $= {}_5C_2 + {}_5C_3 + {}_5C_4 + 5C_5$

Probability $= \dfrac{{}_5C_4}{5C_2 + 5C_3 + 5C_4 + {}_5C_5} = \dfrac{5}{10+10+5+1} = \dfrac{5}{26}$

5. Sample space = $5!$

Successes = $3! \times 2$

Probability $= \dfrac{3! \times 2}{5!} = \dfrac{12}{120} = \dfrac{1}{10}$

6. Sample space = $9 \times 8 \times 7$

Successes = 1

Probability $= \dfrac{1}{9 \times 8 \times 7} = \dfrac{1}{504}$

7. Sample space $= {}_5C_2 \times {}_6C_3 = 200$

Successes $= 1 \times {}_4C_1 \times 1 \times {}_5C_2 = 40$

Probability $= 40/200 = 1/5$

8. Sample space $= 52\,C_5 = 2598960$

Successes $= {}_{12}C_4 \times {}_{40}C_1 = 19800$

Probability $= 0.008$

9. Treat the two T's together as one unit; $\therefore \dfrac{6!}{3!} = 120$ perms.

In total there are $\dfrac{7!}{3!\,2!} = 420$ perms.

$\therefore$ the probability of two T's together is $\dfrac{120}{420} = \dfrac{2}{7}$.

10. If the N's are together, we have $5!/3! = 20$ perms.

In total there are $6!/3!\,2! = 60$ perms.

$\therefore$ the probability of N's together is $20/60 = 1/3$.

$\therefore$ the probability of N's not together is $1 - 1/3 = 2/3$.

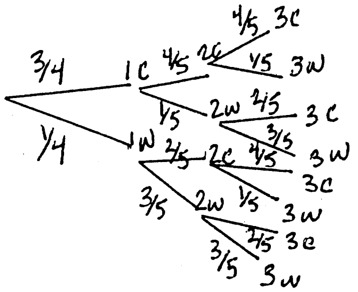
Exercise 44: Probability Using Permutations and Combinations (continued)

11. 6R 4W 5B
a) $P(R) = 6/15 = 2/5$
b) $P(W) = 4/15$
c) $P(B) = 5/15 = 1/3$
d) $P(\overline{R}) = 9/15 = 3/5$
e) $P(R/W) = 10/15 = 2/3$

12. $P(W+A) = 2/9 \times 9/20 = 1/10$

13. $P(3 \text{ or } 5) = 1/6 + 1/6 = 1/3$

14.



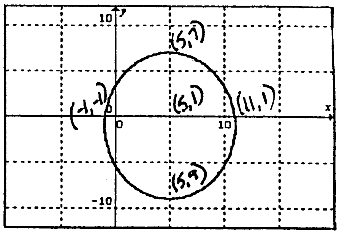
a) $P(2C) = 3/4 \times 4/5 + 1/4 \times 2/5$
$= 14/20 = 7/10$

b) $P(3C) = 3/4 \times 4/5 \times 4/5 + 3/4 \times 1/5 \times 2/5$
$+ 1/4 \times 2/5 \times 4/5 + 1/4 \times 3/5 \times 2/5$
$= 48/100 + 6/100 + 8/100 + 6/100$
$= 68/100$

15. $\dfrac{\tan^3 x + 1}{\tan x + 1} = \sec^2 x - \tan x$

L.H.S $= \dfrac{(\tan x + 1)(\tan^2 x - \tan x + 1)}{\tan x + 1}$
$= \tan^2 x - \tan x + 1$
$= \sec^2 x - 1 - \tan x + 1$
$= \sec^2 x - \tan x$
= R.H.S.

16.



Exercise 44: Probability Using Permutations and Combinations (continued)

17. $\tan^2\theta + 4\sin\theta = \sec^2\theta - 2$

$\tan^2\theta + 4\sin\theta - \sec^2\theta + 2 = 0$

$-1 + 4\sin\theta + 2 = 0$

$4\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{4}$

$\theta_r = 0.253$

III $: \pi + 0.253 = 3.394$

IV $: 2\pi - 0.253 = 6.031$

$\therefore 3.394 + 2k\pi, 6.031 + 2k\pi$ where $k \in I$

18. $A = P\left(1+\frac{r}{n}\right)^{nt}$

$12500 = 8000\left(1+\frac{r}{2}\right)^{2(5)}$

$12500 = 8000\left(1+\frac{r}{2}\right)^{10}$

$\frac{12500}{8000} = \left(1+\frac{r}{2}\right)^{10}$

$\log\left(\frac{12500}{8000}\right) = \log\left(1+\frac{r}{2}\right)^{10}$

$\frac{0.19382}{10} = \frac{10}{10}\log\left(1+\frac{r}{2}\right)$

$0.019382 = \log\left(1+\frac{r}{2}\right)$

$10^{0.019382} = 1+\frac{r}{2}$

$1.0456396 = 1+\frac{r}{2}$

$0\ 0456396 = \frac{r}{2}$

$0.0912791 = r$

$9.13\% = r$

19. Choose 8 students for Biology in ${}_{20}C_8$ ways. Choose 6 from the remaining 12 for physics in ${}_{12}C_6$ ways. The last 6 must take chemistry (No Choice).

$\therefore ({}_{20}C_8)\cdot({}_{12}C_6) = \frac{20!}{8!\,12!}\cdot\frac{12!}{6!\,6!} = 116\ 396\ 280$ ways.

20. $\log_6 92 = \frac{\log 92}{\log 6} = 2.523658$

Exercise 45: Geometric Sequences

1. a) A, P , $d=2$ b) G.P, $r=2$ c) A.P, $d=-3$ d) G.P. , $r=\frac{1}{2}$
e) neither f) G.P. , $r=\sqrt{2}$ g) G.P. $r=-5$

2. $f(1)=3$, $f(2)=3^2=9$, $f(3)=3^3=27$, $f(4)=3^4=81$
This is a G.P. , with $r=3$

3. a) 2, 4, 8 . . . b) 1, 4, 16 . . . c) 6, 18, 54, . . . d) 8, 4, 2 . . .

4. a) $r=2$, $\therefore f(x) = 4(2^{x-1})$, $f(x)=2^{x+1}$, etc.
b) $r=3$, $\therefore f(x) = 6(3^{x-1})$, $f(x) = 2(3^x)$, etc.
c) $r=-2$, $\therefore f(x) = (-2)^{x-1}$, etc.
d) $r=\frac{1}{2}$, $\therefore f(x) = 20(\frac{1}{2})^x$, $f(x) = 10(\frac{1}{2})^{x-1}$, etc.

5. a) $r=2$, $t_1 = 3$, $t_2 = 3(2) = 6$, $t_3 = 3(2)(2) = 3(2^2) = 12$
$t_4 = 3(2^3) = 24$ . . . $t_8 = 3(2^7) = 384$

b) as per the pattern in a) , $t_n = 3(2^{n-1})$

6. a) This is a G.P. with $r = 1.06$.
$t_1 = 10\,000\,(1.06) = \$10\,600$
$t_2 = 10\,000\,(1.06)^2 = 10\,600\,(1.06) = \$11\,236$
$t_3 = 10\,000\,(1.06)^3 = 11\,236\,(1.06) = \$11\,910.16$

b) $\$10\,000(1.06)^n$ c) 12 years
suppose: $10\,000(1.06)^n = 20\,000$
Then $(1.06)^n = 2$
$n \log 1.06 = \log 2$
$$n = \frac{\log 2}{\log 1.06} = 11.9$$
since interest is not credited until the end of the year, it will take 12 years.

Exercise 45: Geometric Sequences (continued)

7. Since a, b and c, from a G.P., let $r = K$; $\therefore$ we have $a = tk$, $b = tk^2$ and $c = tK^3$. Now $\log a$, $\log b$, and $\log c$ become $\log tk = \log t + \log k$, $\log tk^2 = \log t + 2\log k$, and $\log tk^3 = \log t + 3\log k$, which is an A.P. with $d = \log k$.

8. Exponential and logarithmic functions are inverse functions, and since $f(f^{-1}(x)) = x$, $6^{\log_6 17} = 17$.

9. $\log(\cos K) = \dfrac{\log 3 - \log 4}{2}$ or $\log(\cos K) = \frac{1}{2}\log \frac{3}{4}$

$= \log \sqrt{3}/2$

$\log(\cos K) = -0.0624693$ $\therefore \cos K = \sqrt{3}/2$

$\cos K = 10^{-0.0624693} = 0.8660254$ $\therefore K = \pi/6$

$K = 0.524 = \pi/6$

10. There are 36 possible sums. Sums greater than 8 are:
3+6, 4+5, 4+6, 5+4, 5+5, 5+6, 6+3, 6+4, 6+5, 6+6,
$\therefore 10/36 = 5/18$

11. Tan > 0 and cos < 0, $\therefore$ quadrant III
$\cos\theta = -15/17$, $\therefore \theta_R = 0.4899573$, $\therefore \theta = \theta_R + \pi$
and $\csc\theta = \dfrac{1}{\sin\theta} = -2.125$
or
Recognize the 8-15-17 Pythagorean Triple, $\therefore \sin\theta = -8/17$ and $\csc\theta = -17/8$.

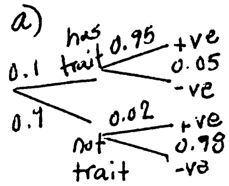
12. a) circle b) hyperbola c) parabola d) line
e) ellipse f) semi-perabola

13. 5 white 3 black
a) $P = P(W+B) + P(B+W)$
$= 5/8 \times 3/7 + 3/8 \times 5/7$
$= 30/56$
$= 15/28$

b) $P = P(W+B) + P(B+W)$
$= 5/8 \times 3/8 + 3/8 \times 5/8$
$= 30/64$
$= 15/32$

Exercise 45: Geometric Sequences (continued)

14. a)



b) $P(+ve) = .1 \times .95 + .9 \times .02$

$= .095 + .018$

$= .113$

c) $P\left(\begin{matrix}\text{Has}\\ \text{trait}\end{matrix} + {+ve}\right) = .1 \times .95 = .095$

d) $P = \dfrac{.1 \times .95}{.1 \times .95 + .9 \times .02}$

$= \dfrac{.095}{.113}$

$= .84$

15. $\dfrac{\sin^3\theta + \csc^3\theta}{\sin\theta + \csc\theta} = \sin^2\theta + \cot^2\theta$

$L.H.S. = \dfrac{\sin^3\theta + \csc^3\theta}{\sin\theta + \csc\theta}$

$= \dfrac{(\sin\theta + \csc\theta)(\sin^2\theta - \sin\theta\csc\theta + \csc^2\theta)}{\sin\theta + \csc\theta}$

$= \sin^2\theta - 1 + \csc^2\theta$

$= \sin^2\theta + \cot^2\theta$

$= R.H.S.$

16. $\log_7(2x+2) - \log_7(x-1) = \log_7(x+1)$

$\log_7(2x+2) - \log_7(x-1) - \log_7(x+1) = 0$

$\log_7 \dfrac{2x+2}{(x+1)(x-1)} = 0$

$7^0 = \dfrac{2x+2}{(x-1)(x+1)}$

$1 = \dfrac{2x+2}{x^2-1}$

$x^2 - 1 = 2x + 2$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ check | $x = -1$ extraneous root

17. $\dfrac{4\cos\theta + 1}{3} - \dfrac{2\cos\theta - 1}{2} = 1$

$8\cos\theta + 2 - 6\cos\theta + 3 = 6$

$2\cos\theta = 1$

$\cos\theta = 1/2$

$\theta = \pi/3$ related angle

$\theta = \pi/3,\ 5\pi/3$

18. $y = 2^{x+1} - 4$

19. $\log_5 \dfrac{a}{b^2} = \log_5 a - \log_5 b^2$ True

$= \log_5 a - 2\log_5 b$

20. a) Choose 6 beads from 10 in ${}_{10}C_6 = \dfrac{10!}{6!4!} = 210$ ways
Permute them in a circle in $5! = 120$ ways. Divide by 2
since it is a bracelet $(210)(120)(1/2) = 12600$

b) Only 4 beads from the 8 are needed ${}_8C_4 = 70$
Permute the 6 beads in 5! & divide by 2

$\therefore \dfrac{70 \times 120}{2} = 4200$

Exercise 46: Geometric Series

1. a) $2+4+6+8+10=30$ b) $1+4+9+16=30$ c) $-2+0+2+4+6=18$

d) $2+4+8+16=30$ e) $2+1+\frac{1}{2}+\frac{1}{4}=\frac{15}{4}$

2. $12(\frac{1}{2}+1+2+4)=90$ $3(2+4+8+16)=90$

Both series are the same.

3. a) $\sum_{K=1}^{4} 3K$ b) $\sum_{n=1}^{8} 6+(n-1)(2)$

$\sum_{n=1}^{8} 6+2n-2$

$\sum_{n=1}^{8} 4+2n$

c) $t_n = t_1 r^{n-1}$

$\therefore \sum_{K=1}^{5} 3(2)^{K-1}$

d) $\sum_{K=1}^{6} (-1)^K K^2$

4. a) arithmetic : A, C b) geometric D, E

5. a) $t_8 = 3(2)^7$

$= 384$

b) $S_8 = \frac{3(1-2^8)}{1-2} = 765$

6 a) $t_1 = 1000$

$t_2 = 1000(1.05) = 1050$

$t_3 = 1050(1.05) = 1102.5$

b) $S_{20} = \frac{1000(1-1.05^{20})}{1-1.05} = 33065.95$

7. a) $S_{10} = \frac{1(1-2^{10})}{1-2}$

$= 1023$

b) $S_{10} = \frac{128(1-\frac{1}{2}^{10})}{1-\frac{1}{2}}$

$= 255.75$

c) $S_{10} = \frac{8(1-1.5^{10})}{1-1.5}$

$= 906.64$

d) $S_{10} = \frac{24(1-\frac{1}{3}^{10})}{1-\frac{1}{3}}$

$= 35.999$ or 36.00

Exercise 46: Geometric Series (continued)

7\. e) $S_{10} = \frac{x^2(1-(x^2)^{10})}{1-x^6} = \frac{x^2(1-x^{20})}{1-x^2}$

8\. first term $8(1.2)^3 = 13.824$

$S_{15} = \frac{13.824(1-1.2^{15})}{1-1.2} = 995.81$

9\. $S_6 = \frac{7.5(1-(\frac{1}{2})^6)}{1-\frac{1}{2}} = 14.77$

10\. a) $2, 3, 4, 5 \rightarrow \frac{4}{13}$

b)

	1	2	3	4	5	6
1	2	3	4	⑤	6	7
2	3	4	⑤	6	7	8
3	4	⑤	6	7	8	9
4	⑤	6	7	8	9	10
5						
6						

$\therefore \frac{4}{36} = \frac{1}{9}$

c) $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

11\. a) $t_1 = 4 \quad t_n = 61$

$\frac{20}{2}(4+61) = 650$

b) t_1 is $t_4 = 2$

$t_{40} = 74$

$S = \frac{37}{2}(2+74) = 1406$

12\. a) independent
b) independent
c) independent
d) dependent

13\. $3(5^{2x-1}) = 75$

$5^{2x-1} = 25$

$5^{2x-1} = 5^2$

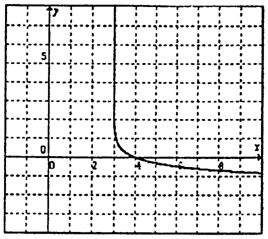
$2x-1 = 2$

$2x = 3$

$x = \frac{3}{2}$

Exercise 46: Geometric Series (continued)

14.



domain $(3, \infty)$
range R
x-int 4
y-int none
asymptote $x=3$

15. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x} = \sin^2 x + \cos^2 x$

$$\frac{\sin x}{\frac{1}{\sin x}} + \frac{\cos x}{\frac{1}{\cos x}}$$

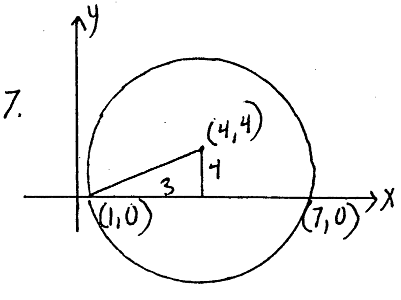
$\sin^2 x + \cos^2 x = \text{RHS}$

16. $3^{\log_2 4} = x$

$3^2 = x$

$9 = x$

17.



$c = (4, 4)$ $R = 5$

Eqn: $(x-4)^2 + (y-4)^2 = 25$

$x^2 + y^2 - 8x - 8y + 7 = 0$

18. a) $\left(\dfrac{1}{2x^2}\right)^8 + 8\left(\dfrac{1}{2x^2}\right)^7(-4x^3) + {}_8C_6\left(\dfrac{1}{2x^2}\right)^6(-4x^3)^2$

$= \dfrac{1}{256x^{16}} - \dfrac{1}{4x^{11}} + \dfrac{7}{x^6}$

b) 5th term ${}_8C_4\left(\frac{1}{2x^2}\right)^4(-4x^3)^4 = \dfrac{8!}{4!\,4!}\left(\dfrac{1}{16x^8}\right)(256x^{12}) = 1120x^4$

19. Choose the face value for the pair in 13 ways.
Choose the pair in ${}_4C_2 = 6$ ways.
Choose the face values for the others in ${}_{12}C_3 = 220$ ways.
Choose each card's suit in 4 ways.
$\therefore 13 \times 6 \times 220 \times 4 \times 4 \times 4 = 1\,098\,240$

20. $\log(2x+1) - \log(x+3) = 0$

$\log \dfrac{2x+1}{x+3} = 0$

$10^0 = \dfrac{2x+1}{x+3}$

$1 = \dfrac{2x+1}{x+3}$

$x + 3 = 2x + 1$

$2 = x$ ✓

Exercise 47: Infinite Geometric Series

1.a) $t_6 = t_1 r^5 = 4(\frac{1}{2})^5 = \frac{1}{8}$

b) $S_6 = \frac{t_1(1-r^6)}{(1-r)} = \frac{4(1-\frac{1}{64})}{1-\frac{1}{2}} = \frac{63}{8} = 7.875$

c) $S_{10} = \frac{t_1(1-r^{10})}{1-r} \quad \frac{4(1-\frac{1}{1024})}{1-\frac{1}{2}} = 7.9922$

d) $S_\infty \frac{t_1}{1-r} = \frac{4}{1-\frac{1}{2}} = 8$

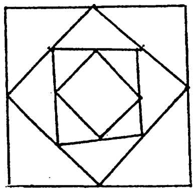
2. a) $S_\infty = \frac{8}{1-\frac{1}{3}} = 8 \cdot \frac{3}{2} = 12$ b) $S_\infty = \frac{16}{1+\frac{1}{2}} = \frac{6}{\frac{3}{2}} = 6 \cdot \frac{2}{3} = 4$

c.) $S_\infty \quad \frac{6}{1-\frac{2}{3}} = 18$ d) $S_\infty = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

3 a) $S_\infty = \frac{8}{1-\frac{1}{2}} = 16$ b) $S_\infty = \frac{2}{1-\frac{1}{3}} = 3$

4. $S_\infty = \frac{4}{1-\frac{1}{4}} = \frac{4}{\frac{3}{4}} = \frac{16}{3}$

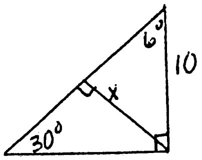
5.



If the large square has perimeter P, each edge of the next square is $\frac{P}{8}\sqrt{2}$. The smaller square has perimeter $4(\frac{P}{8}\sqrt{2}) = \frac{P\sqrt{2}}{2}$.·. $r = \frac{\frac{P\sqrt{2}}{2}}{P} = \frac{\sqrt{2}}{2}$

$S_\infty = \frac{32}{2-\sqrt{2}}$ or $16(\sqrt{2}+2)$

6.



$\sin 60° = \frac{x}{10}$

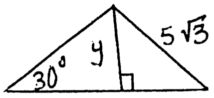
$x = 10 \sin 60°$

$= 10(\frac{\sqrt{3}}{2})$

$= 5\sqrt{3}$

The paths form a geometric sequence with $t_1 = 5\sqrt{3}$ and $r = \frac{\sqrt{3}}{2}$.

$S_\infty = \frac{5\sqrt{3}}{1-\frac{\sqrt{3}}{2}}$



$\sin 60° = \frac{4}{5\sqrt{3}}$

$y = 5\sqrt{3} \sin 60°$

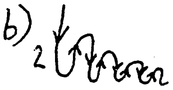
$= 5\sqrt{3}(\frac{\sqrt{3}}{2})$

$= \frac{15}{2}$

$S_\infty = \frac{10\sqrt{3}}{2-\sqrt{3}}$ or $20\sqrt{3}+30$

Exercise 47: Infinite Geometric Series (continued)

7. a) First rise : $2(3/4) = 3/2$
Second rise : $3/2\,(3/4) = 9/8$
Third rise : $9/8\,(3/4) = 27/32$

b)


The "ups" are $\dfrac{\frac{3}{2}}{1-3/4} = 6$.
The "downs" are $2+6 = 8$.
Total distance : 14 meters.

8. a) $S_6 = t_1 \dfrac{(r^b - 1)}{r-1} = \dfrac{6(3^6-1)}{3-1} = 2184$

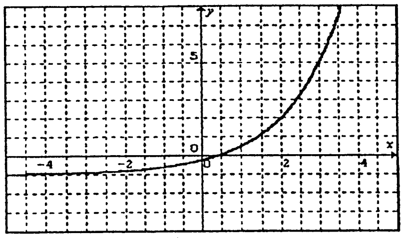
b) $S_9 = \dfrac{t_1(r^9-1)}{r-1} = \dfrac{38.4(1.6^9-1)}{1.6 \cdot 1} = 4334.05$

9. At level n we have reached $1+2+4+\ldots+2^{n-1}$ employees.
This is $\dfrac{1(2n-1)}{2-1} = 2^N-1$ employees.
Thus $2^N-1 \geq 1000$ $2^N \geq 1001$ $N=10$ Since $2^9 = 512$ but $2^{10} = 1024$

10. If a, b, c are arithmetic, then $b-a = c-b$.
Thus $2^{b-a} = 2^{c-b}$
$\dfrac{2^b}{2^a} = \dfrac{2^c}{2^b}$ Thus $2^a, 2^b, 2^c$ is a geometric series.

11. $P(H \text{ or } Q) = P(H) + P(Q) - P(H \cap Q)$
$= 13/52 + 4/52 - 1/52 = 16/52 = 4/13$

12. $P = 1/5 \times 1/4 \times 1/3 \times 1/2 \times 1/1 = \frac{1}{120}$

13.


14 $\log_{81} \dfrac{1}{9} = -\dfrac{1}{2}$

15. $LHS = \dfrac{\sin^2\theta}{1-\sin^2\theta} = \dfrac{\sin^2\theta}{\cos^2\theta} =$
$\tan^2\theta = R.H.S$

16. $P = {}_{20}C_{18}\,(1/5)^{18}(4/5)^2 = \dfrac{3040}{5^{20}} = 3.19 \times 10^{-11}$

Exercise 47: Infinite Geometric Series (continued)

17. $(2^{2x})^3 = 4^{x+3}$

$2^{6x} = 4^{x+3}$

$\log 2^{6x} = \log 4^{x+3}$

$6x \log 2 = (x+3)\log 4$

$6x \log 2 = x \log 4 + 3 \log 4$

$x \log 2^6 = x \log 4 + \log 4^3$

$x \log 64 = x \log 4 + \log 64$

$x(\log 64 - \log 4) = \log 64$

$x \log \frac{64}{4} = \log 64$

$x \log 16 = \log 64$

$x = \frac{\log 64}{\log 16}$

$x = \frac{1.80618}{1.20412}$ $x = 1.5$

or $(2^{2x})^3 = 4^{x+3}$

$2^{6x} = (2^2)^{x+3}$

$2^{6x} = 2^{2x+6}$

$6x = 2x + 6$

$4x = 6$

$x = 3/2$

$x = 1.5$

18. $\sqrt{3} - \tan 4\theta = 0$

$\tan 4\theta = \sqrt{3}$

$4\theta \rightarrow I, III$

$4\theta = 60°, 420°, 780°, 1140°$

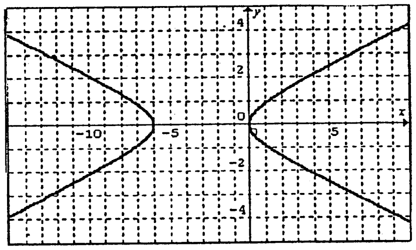
$\theta = 15°, 105°, 195°, 285°$

$4\theta = 240°, 600°, 960°, 1320°$

$\theta = 60°, 150°, 240°, 330°$

$\theta = 15°, 60°, 105°, 150°, 195°,$
$240°, 285°, 330°$

19. $\frac{(x+3)^2}{9} - y^2 = 1$



$x^2 + 6x - 9y^2 = 0$

$(x+3)^2 - 9 - 9y^2 = 0$

$(x+3)^2 - 9y^2 = 9$

$\frac{(x+3)^2}{9} - \frac{y^2}{1} = 1$

20. Solve : $\log_7(x+1) + \log_7(x-5) = 1$

$\log_7(x+1)(x-5) = 1$

$7^1 = (x+1)(x-5)$

$7 = x^2 - 4x - 5$

$0 = x^2 - 4x - 12$

$0 = (x-6)(x+2)$

$x = 6, -2$

But since logs are defined for positive quantities, $x = -2$ must be rejected. $\therefore x = 6$

Exercise 48: Review I

1. Sample space $= {}_{12}C_5$

Successes $= {}_7C_3 \times {}_5C_2$

Probability $= \dfrac{{}_7C_3 \times {}_5C_2}{{}_{12}C_5} = \dfrac{35 \times 10}{792} = \dfrac{350}{792} = \dfrac{175}{396}$

2. $P(T+T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ simultaneously $P = \frac{1}{4}$

3.

S	A ✓
✓✓ T	C ✓

sec θ will be in Quadrant III.

$\csc\theta = -12/7$

$\sin\theta = -7/12$

$\sin^2\theta + \cos^2\theta = 1$

$(-7/12)^2 + \cos^2\theta = 1$

$\dfrac{49}{144} + \cos^2\theta = 1$

$\cos^2\theta = 1 - \dfrac{49}{144}$

$\cos^2\theta = \dfrac{95}{144}$

$\cos\theta = \dfrac{\sqrt{95}}{12}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\sec\theta = \dfrac{1}{\frac{\sqrt{95}}{12}} = \dfrac{12}{\sqrt{95}}$ or

$\dfrac{12\sqrt{95}}{95}$

Since sec θ is in Quad. III, then $\sec\theta = -\dfrac{12\sqrt{95}}{95}$.

4. $P(F \text{ or } H) = P(F) + P(H) - P(F \wedge H)$

$= \dfrac{12}{52} + \dfrac{13}{52} - \dfrac{3}{52} = \dfrac{22}{52} = \dfrac{11}{26}$

Exercise 48: Review I (continued)

5. $\dfrac{\tan x}{1+\sec x} + \dfrac{1+\sec x}{\tan x} = \dfrac{2}{\sin x}$

Proof: $\dfrac{\tan x}{1+\sec x} + \dfrac{1+\sec x}{\tan x}$

$$= \frac{\tan x(\tan x)+(1+\sec x)(1+\sec x)}{(1+\sec x)(\tan x)}$$

$$= \frac{\tan^2 x + 1 + 2\sec x + \sec^2 x}{(1+\sec x)(\tan x}$$

$$= \frac{\sec^2 x - 1 + 1 + 2\sec x + \sec^2 x}{(1+\sec x)(\tan x)}$$

$$= \frac{2\sec^2 x + 2\sec x}{(1+\sec x)(\tan x)}$$

$$= \frac{2\sec x(\sec x + 1)}{(1+\sec x)(\tan x)}$$

$$= \frac{2\sec x}{\frac{\sin x}{\cos x}} = \frac{2}{\frac{\sin x}{\cos x}\cdot\frac{\cos x}{1}}$$

$$= \frac{2}{\sin x}$$

$$\therefore \frac{\tan x}{1+\sec x} + \frac{1+\sec x}{\tan x} = \frac{2}{\sin x} \quad Q.E.D.$$

6. $\log_2 \sqrt[5]{16} = x$

$2^x = \sqrt[5]{16}$

$2^x = 2^{4/5}$

$x = 4/5$

7. $\ln x = \frac{1}{2}\ln 4 + \frac{2}{3}\ln 8$

$\ln x = \ln 4^{1/2} + \ln 8^{2/3}$

$= \ln 2 + \ln 4$

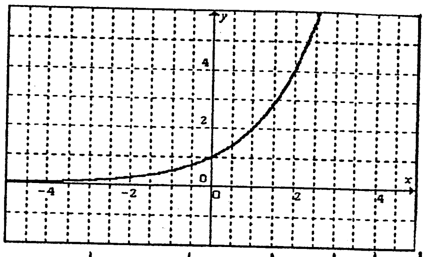
$= \ln(2\cdot 4)$

$\ln x = \ln 8$

$x = 8$ ✓ check

L.S. = R.S.

8.



X	-3	-2	-1	0	1	2
y	0.125	0.25	0.5	1	2	3

9. $\log_5 \dfrac{\sqrt[7]{AB^3}}{(CD)^2}$

10. $\log_6\left(\frac{50}{27}\right) = \log_6 50 - \log_6 27$

$= \log_6(25\cdot 2) - \log_6 3^3$

$= \log_6 5^2 + \log_6 2 - 3\log_6 3$

$= 2\log_6 5 + \log_6 2 - 3\log_6 3 = 2r+p-3q$

B

11. note: if $y = f(|x|)$

the $f(x) = f(x)$

graph represents even function

12. $S_\infty = \dfrac{t_1}{1-r}$

$30 = \dfrac{t_1}{1-\frac{1}{3}}$

$(30)\left(\frac{2}{3}\right) = t_1$

$t_1 = 20$

Exercise 48: Review I (continued)

13. $2(x^2-3x)+3(y^2+6y)=12$

$2(x^2-3x+\frac{9}{4})+3(y^2+6y+9)=12+\frac{9}{2}+27$

$2(x-\frac{3}{2})^2+3(y+3)^2=\frac{87}{2}$

Centre : $(\frac{3}{2}, -3)$

14. a) $\frac{6!}{3!}$ = 120 arrangements

b) The 4 middle letters are CHSE which can be arranged in 4! ways. The probability of beginning and ending with E is $\frac{4!}{120}=\frac{1}{5}$.

15. Period: 4π $B=\frac{2\pi}{4\pi}=\frac{1}{2}$ $y=A\cos B(x+c)+D$ $y=A\sin B(x+c)+D$

Amplitude $4=A$ $A=4$

$c=D=0$ $y=4\cos(\frac{1}{2}(x))$ $B=\frac{1}{2}$

Equation: $y=4\cos\frac{x}{2}$ $C=3\pi$

Other possibilities exist. or $y=4\sin\frac{(x-3\pi)}{2}$ $D=0$

$=4\sin(\frac{1}{2}(x-3\pi))$

16. No. I disagree. The value of $|r|$ is $+\frac{5}{4}$, so the series does not converge.

17. a) $\frac{3}{5}(360)=3(72)=216^\circ$

b) $\frac{3}{5}(2\pi)=\frac{6\pi}{5}$ radians

Exercise 48: Review I (continued)

18. $\sin\alpha = 3/5$ $\qquad \sin\beta = 5/13$

$\cos\alpha = 4/5$ $\qquad \cos\beta = 12/13$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(\tfrac{4}{5}\right)\left(\tfrac{12}{13}\right) - \left(\tfrac{3}{5}\right)\left(\tfrac{5}{13}\right)\\ &= \frac{48-15}{65} = \frac{33}{65}\end{aligned}$$

19.

$$\frac{\overset{\substack{\text{choose 2}\\ \text{ranks for}\\ \text{pairs}}}{\downarrow}{}_{13}C_2 \cdot \overset{\substack{\text{choose}\\ \text{a pair}}}{\downarrow}{}_{4}C_2 \cdot \overset{\substack{\text{choose}\\ \text{a pair}}}{\downarrow}{}_{4}C_2 \cdot \overset{\substack{\text{choose the}\\ \text{rank for 3}\\ \text{of a kind}}}{\downarrow}{}_{11}C_1 \cdot \overset{\substack{\text{choose 3 of}\\ \text{the 4 cards}}}{\downarrow}{}_{4}C_3}{{}_{52}C_7}$$

20. $\left(\tan\frac{\pi}{3} + \cos\frac{\pi}{6}\right)\left(\sin\frac{7\pi}{3}\right)$

$$\begin{aligned}&= \left(\sqrt{3} + \frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{3}{2}(\sqrt{3}) \cdot \frac{1}{2}\sqrt{3}\\ &= \frac{9}{4}\end{aligned}$$

Exercise 49: Review II

1a) (H,H,H,H), (HHHT), (HHTH), (HTHH), (THHH), (HHTT), (HTHT), (THTH), (TTHH), (THHT), (HTTH), (TTTH), (TTHT), (THTT), (HTTT), (TTTT)

16 ways

b) P(at least 3 tails) = $5/16$

2. $\dfrac{\tan x+1}{1-\tan x} = \dfrac{\sin x+\cos x}{\cos x-\sin x}$

$LHS = \dfrac{\frac{\sin x}{\cos x}+1}{1-\frac{\sin x}{\cos x}}$

$= \dfrac{\frac{\sin x+\cos x}{\cos x}}{\frac{\cos x-\sin x}{\cos x}}$

$= \dfrac{\sin x+\cos x}{\cos x-\sin x}$

$= RHS$

3. sample $= {}_{52}C_4$

face cards J, Q, K only $3\times 4=12$

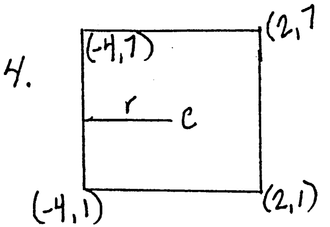
$= {}_{12}C_2$

other two cards $= {}_{40}C_2$

$P(\text{2 face}) = \dfrac{{}_{12}C_2 \cdot {}_{40}C_2}{{}_{52}C_4}$

$= \dfrac{66 \cdot 780}{270725} = 0.19$

4.



$C\left(\dfrac{-4+2}{2}, \dfrac{1+7}{2}\right)$ $r=|-4 - -1|$

$r=3$

$C(-1, 4)$

$(x+1)^2+(y-4)^2=9$

$x^2+2x+1+y^2-8y+16=9$

$x^2+y^2+2x-8y+8=0$

5. $\log_5(x^2+2x+5)-\log_5(x-5)=2$

$\log_5\left(\dfrac{x^2+2x+5}{x-5}\right)=2$

$5^2 = \dfrac{x^2+2x+5}{x-5}$

$25x-125 = x^2+2x+5$

$0 = x^2-23x+130$

$0=(x-13)(x-10)$

$x=13,\ x=10$

check: both are acceptable values

Exercise 49: Review II (continued)

6. $\tan^2\theta = \tan\theta$

$\tan^2\theta - \tan\theta = 0$

$\tan\theta(\tan\theta - 1) = 0$

$\tan\theta = 0 \quad \tan\theta = 1$

$\theta = \{0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}\}$

7. $\tan^2\theta - \sec\theta - 1 = 0$

$\sec^2\theta - 1 - \sec\theta - 1 = 0$

$\sec^2\theta - \sec\theta - 2 = 0$

$(\sec\theta - 2)(\sec\theta + 1) = 0$

$\sec\theta = 2$ or $\sec\theta = -1$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi, \pi + 2k\pi$

where $k \in I$

8. $c(0,0)$ $r = 3$

$x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9 - x^2}$

$y = +\sqrt{9 - x^2}$

9. $f(x) = x(x^2+1)^{\frac{1}{2}}$

$\ln[f(x)] = \ln[x(x^2+1)^{\frac{1}{2}}]$

$\ln[f(x)] = \ln x + \frac{1}{2}\ln(x^2+1)$

10. $2\sin^2\theta - \sin\theta = 3$

$2\sin^2\theta - \sin\theta - 3 = 0$

$(2\sin\theta - 3)(\sin\theta + 1) = 0$

$\sin\theta = \frac{3}{2} \quad \sin\theta = -1$

no soln.

$\theta = \{\frac{3\pi}{2} + 2k\pi, k \in I\}$

11. sample $6 \cdot 6 = 36$

sum of 8 $(2,6)(3,5)(4,4)(5,3)$ $(6,2)$

$P(\text{sum} = 8) = \frac{5}{36}$

12. $(\frac{1}{2})^x = 8$

$2^{-x} = 2^3$

$-x = 3$

$x = -3$

13. a) $P(5) = \frac{1}{6} \cdot P(4) = \frac{1}{6}$

$P(5 \cap 4) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

b) $P(5 \cap 4) = \frac{1}{36}$, $P(4 \cap 5) = \frac{1}{36}$

$P(5 \cap 4)$ or $P(4 \cap 5) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$

14. $\log_{\frac{1}{2}} 32$

$x = \log_{\frac{1}{2}} 32$

$(\frac{1}{2})^x = 32$

$2^{-x} = 2^5$

$-x = 5$

$x = -5$

Exercise 49: Review II (continued)

15. $\tan\alpha + \cot\alpha \equiv \sec\alpha\csc\alpha$

$LHS \equiv \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\alpha}{\sin\alpha}$

$\equiv \dfrac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha\ \sin\alpha}$

$\equiv \dfrac{1}{\cos\alpha\ \sin\alpha}$

$\equiv \dfrac{1}{\cos\alpha} \cdot \dfrac{1}{\sin\alpha}$

$\equiv \sec\alpha \cdot \csc\alpha$

$\equiv RHS.$

solution 1:

16. $3^{\log_3 x} = 4$

$\log_3 x \cdot \log_3 3 = \log_3 4$

$\log_3 x = \log_3 4$

solution 2: $x = 4$

$3^{\log_3 x} = 4$

$\log_3 4 = \log_3 x$

solution 3: $4 = x$

if $e^{enx} = x$

it follows that $3^{\log_3 x} = x$

$\therefore x = 4$

17. $\log_5 \left[\dfrac{x^2(1-5x)^{3/2}}{(x^3-x)^{1/2}}\right]$

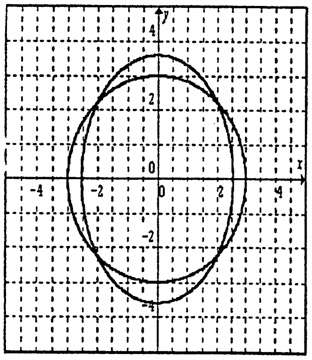
$= \log_5 x^2 + \log_5 (-5x)^{3/2} - \log_5 (x^3-x)^{1/2}$

$= 2\log_5 x + \frac{3}{2}\log_5(1-5x) - \frac{1}{2}\log_5(x^3-x)$

$= 2\log_5 x + \frac{3}{2}\log_5(1-5x) - \frac{1}{2}\log_5[x(x-1)(x+1)]$

$= 2\log_5 x + \frac{3}{2}\log_5(1-5x) - \frac{1}{2}[\log_5 x + \log_5(x-1) + \log_5(x+1)]$

18. a)



$x^2 + y^2 = 9$ circle, $r = 3$

$2x^2 + y^2 = 13$

$\dfrac{2x^2}{13} + \dfrac{y^2}{13} = 1$

$\dfrac{x^2}{13/2} + \dfrac{y^2}{13} = 1$ ellipse

$a = \sqrt{13} = 3.6$

$b = \sqrt{13/2} = 2.5$

b) $2x^2 + y^2 = 13$

$x^2 + y^2 = 9$

$x^2 = 4$

$x = \pm 2$

$x = 2$ $(2)^2 + y^2 = 9$

$y^2 = 5$

$y = \pm\sqrt{5}$

$x = -2$ $(-2)^2 + y^2 = 9$

$y^2 = 5$

$y = \pm\sqrt{5}$

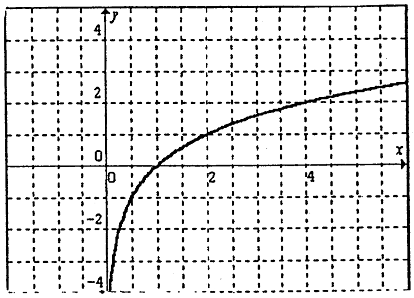
$\{(2, \sqrt{5}), (-2, \sqrt{5}), (2, -\sqrt{5}), (-2, -\sqrt{5})$

Exercise 49: Review II (continued)

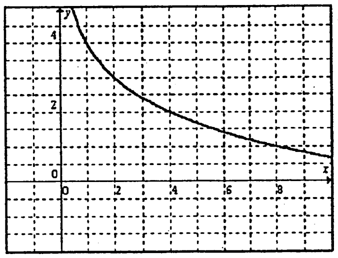
19. a) tie the friends together
   9 'people' in 9! ways
   3 friends in 3! ways
   $9! \times 3! = 2177280$

b) tie friends together
   9 'people' in $(9-1)! = 8!$ ways
   3 friends in 3! ways
   $8! \times 3! = 241920$

20. a) $y = \log_2 x$
   $2^y = x$



b) $y = 4 - \log_2 x$
→ flip over x-axis,
move up 4 units



21. $\sin\theta = -\frac{\sqrt{7}}{3}$

$\cos\theta = \frac{-2}{3}$

$\sin(\theta + \pi) = \sin\theta \cos\pi + \cos\theta \sin\pi$

$= \left(\frac{-\sqrt{7}}{3}\right)(-1) + \left(\frac{-2}{3}\right)(0)$

$= \frac{\sqrt{7}}{3}$

Exercise 50: Review III

1. $\sum_{k=3}^{10} \frac{2^4}{2^{k-6}} = (24)(8)+(24)(2)+\cdots+\frac{(24)}{16}$

$S_8 = \frac{(24)(8)\left[1-\left(\frac{1}{2}\right)^8\right]}{1-\frac{1}{2}} = 382.5$

2. $\sqrt{2x+1} = 1+\sqrt{x+4}$

$2x+1 = 1+2\sqrt{x+4}+x+4$

$x-4 = 2\sqrt{x+4}$

$x^2-8x+16 = 4x+16$

$x^2-12x = 0$

$x(x-12) = 0$

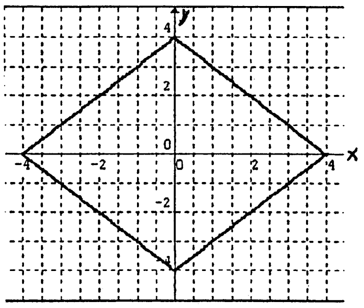
Possible answers are $x=0$, $x=12$.

Check : if $x=0$, $\sqrt{1}-\sqrt{4}=1$ Reject

if $x=12$, $\sqrt{25}-\sqrt{16}=1$ ✓

$\therefore x=12$

3.



4. $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$

a) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{7}}{4}\right)\left(\frac{-3}{4}\right) = \frac{-3\sqrt{7}}{8}$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9}{16}-\frac{7}{16} = \frac{1}{8}$

5. $y = 8-x$

$\therefore (x-1)^2+(8-x)^2=25$

$x^2-2x+1+64-16x+x^2=25$

$2x^2-18x+40=0$

$x^2-9x+20=0$

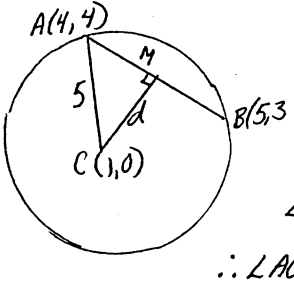
$(x-4)(x-5)=0$

$x=4$ or $x=5$

The points are $(4,4)$ and $(5,3)$

6. The centre is $(1,0)$ and the radius is 5.

Solution One:



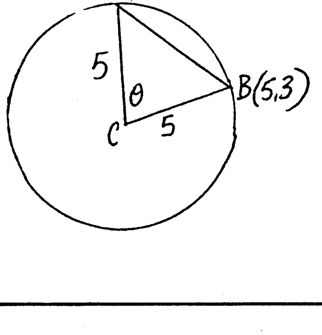
$d = \frac{|1+0-8|}{\sqrt{1+1}} = \frac{7}{2}$

$\cos\angle MCA = \frac{\frac{7}{2}}{5}$

$\angle MCA = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) = 8.13°$

$\therefore \angle ACB = 2(8.13°) = 16.3°$

6. Solution Two:



$AB = \sqrt{(4-5)^2+(4-3)^2} = \sqrt{2}$

By the law of cosines:

$(\sqrt{2})^2 = 5^2+5^2-2(5)(5)\cos\theta$

$2 = 50-50\cos\theta$

$\cos\theta = \frac{48}{50} = \frac{24}{25} =$

$= 0.96$

$\theta = \cos^{-1}(0.96)$

$= 16.3°$

Exercise 50: Review III (continued)

7. $D = D_0 e^{kt}$

$15.62 = D_0 e^{2k}$ $\qquad$ $21.724 = D_0 e^{5k}$

$\frac{15.62}{e^{2k}} = D_0$ $\qquad$ $\frac{21.724}{e^{5k}} = D_0$

$\frac{15.62}{e^{2k}} = \frac{21.724}{e^{5k}}$

$21.724 = \frac{15.62e^{5k}}{e^{2k}}$

$\frac{21.724}{15.62} = \frac{15.62\,e^{3k}}{15.62}$

$1.390781 = e^{3k}$

$\ln 1.390781 = \ln e^{3k}$

$\ln 1.390781 = 3k$

$0.3298655 = 3k$

$0.11 = k$

$D_0 = \frac{15.62}{e^{2k}} = \frac{15.62}{e^{2(0.11)}} = \frac{15.62}{e^{0.22}} = 12.54$ cm

or $21.724 = 15.62\,e^{3k}$

$\ln(2.724) = \ln 15.62 + \ln e^{3k}$

$\ln 21.724 - \ln 15.62 = 3k$

$k = \frac{\ln 21.724 - \ln 15.62}{3}$

$k = 0.10996$

$21.727 = D_0 e^{(0.10996)(5)}$

$D_0 = 12.54$ cm

8. $\log_5 200 = \frac{\ln 200}{\ln 5} = 3.2920$

9. Let $x = 453^{62}$

$\log x = 62 \log 453$

$= 164.67$

thus $10^{164} < x < 10^{165}$

x will have 165 digits

10. $\sin\theta + 2\sin\theta\cos\theta = 0$

$\sin\theta(1 + 2\cos\theta) = 0$

$\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

11. $(2x - \frac{1}{2}y)^7 = (2x)^7 + 7(2x)^6(-\frac{1}{2}y) + 21(2x)^5(-\frac{1}{2}y)^2 \ldots$

$= 128x^7 - 224x^6y + 168x^5y^2$

12. It can be bgbgbg or gbgbgb. That decision can be made in 2 ways. The boys can be arranged in 3! = 6 ways.
" girls " " " " = 6 " .
There are (2)(6)(6) = 72 ways.

Exercise 50: Review III (continued)

13. $$\frac{3n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1} = \frac{5(n-1)(n-2)(n-3)(n-4)(n-5)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$3n = \frac{5(n-4)(n-5)}{5} \qquad 3n = n^2 - 9n + 20$$

$$0 = n^2 - 12n + 20$$

$$0 = (n-2)(n-10) \quad n > 2 \therefore n = 10$$

14. a) $$10 = 16r^2 \quad r^2 = \frac{10}{16} = \frac{5}{8} \quad r = \frac{\sqrt{5}}{2\sqrt{2}} \text{ or } \sqrt{5/8} = 0.79057 \quad r = \frac{\pm\sqrt{5}}{2\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\pm\sqrt{10}}{4}$$

b) $$16 = t_1 r^2 \quad t_1 = \frac{16}{r^2} = 16\cdot\frac{8}{5} = 128/5$$

$$S_\infty = \frac{t_1}{1-r} = \frac{\frac{128}{5}}{1-\frac{\sqrt{5}}{2\sqrt{2}}} = \frac{25.6}{1-0.79057} = 122.24$$

for $r = -\frac{\sqrt{10}}{4}$ $$S_\infty = \frac{t_1}{1-r} = \frac{\frac{128}{5}}{1+\frac{\sqrt{10}}{4}} = \frac{\frac{128}{5}}{\frac{4+\sqrt{10}}{4}} = \frac{128}{5}\cdot\frac{4}{4+\sqrt{10}} = 14.30$$

15. $$\log_a\sqrt{x} = \tfrac{1}{2}\log_a x = \tfrac{1}{4}(2\log_a x) = \tfrac{1}{4}\log_a x^2 = \frac{0.6}{4} = 0.15$$

16. Let $b = ar$

then $c = ar^2$

$abc = a(ar)(ar^2) = a^3r^3$

$a^3r^3 = 8$

$ar = 2$

$b = 2$ It is not possible to find a or c.

17. $$[(2^3)^x]^2(2^2)^x = \frac{\sqrt{2^x}}{2^4}$$

$$2^{6x}2^{2x} = \frac{\sqrt{2^x}}{2^4}$$

$$2^{8x} = 2^{x/2-4}$$

$$8x = \frac{x}{2} - 4$$

$$16x = x - 8 \qquad x = -\frac{8}{15}$$

$$15x = -8$$

Exercise 50: Review III (continued)

18. Case 1: All letters different.

$$5 \cdot 4 \cdot 3 \cdot 2 = 120$$

Case 2: One pair (Either A or N). The rest different:

$$(2)({}_4C_2)\frac{4!}{2!} = 2(6)(12) = 144$$

Case 3: Two pairs: AA, NN

$$\frac{4!}{2!2!} = 6$$

Case 4: Three A's

$$_4C_1\left(\frac{4!}{3!}\right) = 16$$

$$\begin{array}{r} 120 \\ 144 \\ 6 \\ 16 \\ \boxed{286} \end{array}$$

Total = 286

19. Note $\log_p q = \frac{1}{\log_q p}$ This is easily proved from the change of base formula.

So, $\log_{ab} M = \frac{1}{\log_m ab} = \frac{1}{\log_m a + \log_m b}$

$$= \frac{1}{\frac{1}{\log_a m} + \frac{1}{\log_b m}} = \frac{1}{\frac{1}{x} + \frac{1}{y}} = \frac{xy}{x+y}$$

20. $\frac{2\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta - 2\sin^2\theta}{2\sin\theta\cos\theta}$

$$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta} - \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \cot\theta - \tan\theta$$