The Factor Theorem

What does it mean if $P(a) = 0$?

$$\boxed{P(x) \text{ has the factor } (x - a) \text{ if and only if } P(a) = 0}$$

Which of the following binomials are factors of $P(x) = x^3 - 3x^2 - x + 3$?

a) $x - 1$
b) $x + 3$

$P(1) = (1)^3 - 3(1)^2 - (1) + 3$
$= 0$

$P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 3$
$= -27 - 27 + 3 + 3$
$= -48$

$\therefore (x-1)$ is a factor

ex. Given that (x + 2) is a factor of P(x), express P(x) as a product of its factors.

$P(x) = x^3 + 2x^2 - x - 2$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -1 & -2 \\ & \downarrow & -2 & 0 & 2 \\ \hline & 1 & 0 & -1 & \underbrace{0}_{R} \end{array}$$

$P(x) = (x^2 - 1)(x + 2)$ ← stop here
$= (x+1)(x-1)(x+2)$ ← if fully factored

ex. Given $x = 4$ $(x-4)$ is one of the zeroes of $Q(x) = 2x^3 - 5x^2 - 11x - 4$
determine all of the other zeroes.

$$\begin{array}{r|rrrr} 4 & 2 & -5 & -11 & -4 \\ & \downarrow & 8 & 12 & 4 \\ \hline & 2 & 3 & 1 & \underbrace{0}_{R} \end{array}$$

$$Q(x) = (2x^2 + 3x + 1)(x-4)$$
$$0 = (2x+1)(x+1)(x+4)$$
$$x = -\tfrac{1}{2}, -1, 4$$

ex. Fully factor: $M(x) = 2x^3 - 9x^2 + 7x + 6.$

± 1
± 2
± 3
± 6

$$M(-1) = 2(-1)^3 - 9(-1)^2 + 7(-1) + 6$$
$$= -2 - 9 - 7 + 6$$
$\therefore (x+1)$ is not a factor

$$M(+2) = 2(+2)^3 - 9(+2)^2 + 7(+2) + 6$$
$$= 16 - 36 + 14 + 6$$
$$= 0$$

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 7 & 6 \\ & \downarrow & 4 & -10 & -6 \\ \hline & 2 & -5 & -3 & \underbrace{0}_{R} \end{array}$$

$$M(x) = (2x^2 - 5x - 3)(x-2)$$
$$= (2x+1)(x-3)(x-2)$$

ex. When P(x) is divided by x - 3, it has a quotient of $2x^2 + x - 6$ and a remainder of 4. Determine P(x).

$$\frac{P(x)}{(x-3)} = (2x^2+x-6) + \frac{4}{(x-3)}$$

stop here if not asking to simplify → $P(x) = (2x^2+x-6)(x-3) + 4$

dividend quotient divisor remainder

ex. When $2x^3 + kx^2 - 3x + 2$ is divided by x - 2, the remainder is 4. Determine the value of k.

$2(2)^3 + k(2)^2 - 3(2) + 2 = 4$

$16 + 4k - 6 + 2 = 4$

$4k = -8$

$k = -2$

$$\begin{array}{r|rrrr} 2 & 2 & k & -3 & 2 \\ & \downarrow & 4 & 2k+8 & 4k+10 \\ \hline & 2 & k+4 & 2k+5 & \underbrace{4k+12}_{4} \end{array}$$

$4k + 12 = 4$

$4k = -8$

$k = -2$

ex. Find the value of a if (x - 2) is a factor of $ax^3 + 4x^2 + x - 2$.

$$a(2)^3 + 4(2)^2 \cancel{+ (2) - 2} = 0$$
$$8a + 16 = 0$$
$$8a = -16$$
$$a = -2$$

$$\begin{array}{c|cccc} 2 & a & 4 & 1 & -2 \\ & \downarrow & 2a & 4a+8 & 8a+18 \\ \hline & a & 2a+4 & 4a+9 & 8a+16 \end{array}$$

$= 0$

$$8a + 16 = 0$$
$$8a = -16$$
$$a = -2$$